THE LINEAR QUADRATIC OPTIMAL CONTROL PROBLEM FOR INFINITE DIMENSIONAL SYSTEMS OVER AN INFINITE HORIZON; SURVEY AND EXAMPLES*

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Abstract

Survey of currently available theory for systems the evolution of which can be described by semigroups of operators of class C_0. Connection between the concepts of stabilizability and detectability and the problem of existence and uniqueness of solutions to the operator Riccati equation. Examples and open problems.

1. Introduction.

For systems described by ordinary differential equations the infinite-time quadratic cost problem is well-studied (cf. R. BROCKETT [1], R.E. KALMAN [1],[2], J.C. WILLEMS [1], W.M. WONHAM [1]). This problem has been studied for certain classes of infinite-dimensional systems. J.L. LIONS [1] has studied this problem for abstract evolution equations of parabolic type and given a complete solution to the problem. LUKES-RUSSELL [1] have studied this problem for abstract evolution equations of the type

$$(1) \qquad \frac{dx}{dt} = Ax + Bu, \quad x(0) = x_0 \in \mathcal{D}(A),$$

where A is an unbounded spectral operator (cf. DUNFORD-SCHWARTZ [1]) and B is also an unbounded operator satisfying certain conditions. LUKES-RUSSELL [1] also allow unbounded operators in the cost function. Using an approach originally due to R.E. KALMAN [2] they obtain an operational differential equation of Riccati type to characterize the time-varying feedback gain in the finite time case. They also show that under an appropriate stabilizability hypothesis the solution to the infinite-time quadratic cost problem can be obtained in feedback form, where the "feedback gain" is characterized by the solution of an operator equation of quadratic type. The same problem has also been studied by R. DATKO [5]. Unfortunately R. DATKO [5] does not characterize the solution as a feedback controller acting on the "state" of the system.

The objective of this paper is to survey available results for a special class of infinite dimensional control systems the evolution of which is characterized by a semi-group of operators of class C_0. The approach we use here is different from that of LUKES-RUSSELL [1] as well as R. DATKO [3] and constitutes a synthesis of the work of J.L. LIONS [1] and DELFOUR-McCALLA-MITTER [1]. Complete results and detailled argument are to be found in a forthcoming monograph (cf. BENSOUSSAN-DELFOUR-MITTER [1]). Our results also make use of the work of R. DATKO [2] on Stability Theory in Hilbert spaces and J. ZABCZYK [1] on the concept of detectability in Hilbert spaces. In doing this, we insist on an approach which clarifies the system-theoretic relationship between controllability, stabilizability, stability and existence of a solution of an associated operator equation of Riccati type.

This theory covers certain classes of distributed controls; it also covers hereditary systems which can be looked as distributed parameter system with boundary control. At this time, it does not seem possible to systematically deal with boundary control problems. However, in a different framework, results are now available (cf. D.L. RUSSELL [6]).

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Notation

Let X and Y be two real Hilbert spaces with norms $|\ |_X$ and $|\ |_Y$ and inner product $(\ ,\ )_X$ and $(\ ,\ )_Y$. The space of all continuous linear maps $T : X \to Y$ endowed with the natural norm

$$|T| = \sup\{|Tx|_Y : |x|_X \le 1\}$$

will be denoted $\mathcal{L}(X,Y)$. When X=Y we shall use the notation $\mathcal{L}(X)$. The transposed operator of T in $\mathcal{L}(X,Y)$ is an element of $\mathcal{L}(Y',X')$ which will be denoted T*, where X' and Y' are the topological dual of X and Y. T in $\mathcal{L}(X)$ is self-adjoint if T*=T; a self-adjoint operator T is positive, $T \ge 0$, if for all x in X $(x,Tx) \ge 0$.

2. Preliminaries and problem formulation.

Let X, U and Y be real Hilbert spaces. Let B be an element of $\mathcal{L}(U,X)$ and let A be an unbounded closed operator on X with domain $\mathcal{D}(A)$. We assume that A is the infinitesimal generator of a strongly continuous semi-group $\{S(t) : t \ge 0\}$ of class C_0. We denote by $\mathcal{D}(A)$ the domain of A endowed with the graph norm

$$(2.1) \qquad \|v\|^2 = |v|_2^2 + |Av|_2^2.$$

We write $i : V \to X$ the continuous dense injection of

V into X. We also introduce the topological duals V' and X' of V and X, respectively. We identify elements of X and X' and denote by i* the adjoint map of i:

$$(2.2)\qquad V \xrightarrow{i} X \equiv X' \xrightarrow{i^*} V'.$$

We now consider the system

$$(2.3)\qquad \dot{x} = Ax + Bv \text{ in } [0,\infty[, \quad x(0) = x_0 \in V$$

or more generally

$$(2.4)\qquad x(t) = S(0)x_0 + \int_0^t S(t-s)Bv(s)ds.$$

Equation (2.4) can be looked at as a "weak solution" of equation (2.3) and any solution of (2.3) will be of the form (2.4).

We associate with the control function v and the trajectory x the cost function

$$(2.5)\qquad J(v,x_0) = \int_0^\infty [|Hx(t)|_Y^2 + (Nv(t),v(t))_U]dt,$$

where H and N belong to $\mathcal{L}(X,Y)$ and $\mathcal{L}(U)$, respectively. Moreover there exists a constant c > 0 such that

$$(2.6)\qquad \forall\, v,\quad (Nv,v)_U \geq c|v|_U^2.$$

Given x^0, the optimal control problem consists in minimizing the cost function (2.5) over all v in $L^2_{loc}(0,\infty;U)$

$$(2.7)\qquad \operatorname{Inf}\{J(v,x_0) : v \in L^2_{loc}(0,\infty;U)\}.$$

3. Asymptotic behaviour and L^2-stability.

In order to make sense of problem (2.7), it is necessary to introduce concepts of stability to characterize the asymptotic behaviour of solutions of system (2.3) as the time t goes to infinity. Consider the uncontrolled system

$$(3.1)\qquad \dot{x} = Ax \text{ in } [0,\infty[, \quad x(0) = x_0$$

with observation

$$(3.2)\qquad y(t) = Hx(t).$$

Definition 3.1. (i) A is said to be L^2-stable with respect to H if

$$(3.3)\qquad \forall\, x_0,\quad \int_0^\infty |Hx(t)|^2 dt < \infty.$$

(ii) A is said to be L^2-stable if it is L^2-stable with respect to the identity I with Y=X.

(iii) The pair (A,B) is said to be stabilizable with respect to H if

$$(3.4)\qquad \forall\, x_0,\ \exists\, v \in L^2(0,\infty;U) \text{ such that } \int_0^\infty |Hx(t)|_Y^2 dt < \infty.$$

(iv) The pair (A,B) is said to be stabilizable if it is stabilizable with respect to the identity I with Y=X. □

Theorem 3.2. The following statements are equivalent:

(i) A is L^2-stable with respect to H;

(ii) There exists an element B of $\mathcal{L}(X)$ such that

$$(3.5)\qquad \forall\, x,y \in X,\ (Bx,y) = \lim_{t\to\infty} \int_0^t (H\Lambda(t)x, H\Lambda(t)y)_Y dt;$$

(iii) There exists a positive self-adjoint element D of $\mathcal{L}(X)$ such that

$$(3.6)\qquad A^*Di + i^*DA + i^*H^*H = 0 \text{ in } \mathcal{L}(V,V').$$

Proof: Cf. R. DATKO [2], DELFOUR-McCALLA-MITTER [1], BENSOUSSAN-DELFOUR-MITTER [1]). □

Corollary 1. If D in $\mathcal{L}(X)$ is a positive self-adjoint solution of (3.6), then D ≥ B. Moreover for all x in X, the map $t \to (\Lambda(t)x, D\Lambda(t)x)$ is a monotonically decreasing function of t and for all x and y in X

$$(3.7)\qquad \lim_{t\to\infty} (\Lambda(t)x, D\Lambda(t)y) = (x,Dy)-(x,By). \quad □$$

Corollary 2. Any of the statements in Theorem 3.2 implies that

$$(3.8)\qquad \begin{aligned} &\forall\, x \in V,\ \lim_{t\to\infty} HS(t)x = 0, \text{ and,} \\ &\forall\, x \in X,\ \lim_{t\to\infty} BS(t)x = 0. \quad □ \end{aligned}$$

The above results are specialized in the following theorem on L^2-stability.

Theorem 3.3. The following statements are equivalent.

(i) A is L^2-stable;

(ii) There exists an element B of $\mathcal{L}(X)$ such that

$$(3.9)\qquad \forall\, x,y \in X,\ (Bx,y) = \lim_{t\to\infty} \int_0^t (S(t)x,S(t)y)dt;$$

(iii) There exists a positive self-adjoint element D of $\mathcal{L}(X)$ such that

$$(3.10)\qquad A^*Di + i^*DA + i^*i = 0 \text{ in } \mathcal{L}(V,V');$$

(iv) The type ω_0 of the semi-group $\Lambda(t)$ is strictly negative;

(v) There exist μ < 0 and M ≥ 1 such that for all x in X and t ≥ 0

$$(3.11)\qquad |S(t)x| \leq M \exp(\mu t)|x|.$$

(vi) $\lim_{t\to\infty} S(t) = 0$ in $\mathcal{L}(X)$. □

4. The optimal control problem in $[0,\infty[$

In order to make sense of problem (2.7) we must check that for each x_0 there exists at least one v in $L^2_{loc}(0,\infty;U)$ such that the cost $J(v,x_0)$ be finite. This is precisely the stabilizability of the pair (A,B) with respect to H.

Theorem 4.1. Let (A,B) be stabilizable with respect to H, then there exists a unique control function u in $L^2_{loc}(0,\infty;U)$ which minimizes $J(v,x_0)$ for a given x_0 over all elements of $L^2_{loc}(0,\infty;U)$. Moreover this control function u can be synthesized via the feedback law

$$(4.1) \qquad u(t) = -N^{-1}B^*\Pi x(t),$$

where Π is a positive self-adjoint element of $\mathcal{L}(X)$. The transformation Π of X is also a solution of the operator Riccati equation

$$(4.2) \quad A^*\Pi i + i^*\Pi A + i^*[H^*H-\Pi R\Pi]i = 0, \text{ in } \mathcal{L}(V,V'),$$

$$R = B^*N^{-1}B. \quad \square$$

In general Π will not be the unique positive self-adjoint solution of (4.2) and the closed-loop system

$$(4.3) \qquad \dot{x} = [A-R\Pi]x, \quad x(0) = x_0$$

will not be L^2-stable.

Nevertheless a sufficient condition is available in order to answer those two questions. It makes use of the classical concept of detectability as introduced by W.M. WONHAM [1].

Definition 4.2. The pair (A,H) is said to be detectable if the pair $(A^\dagger,H^*)$ is stabilizable ($A^\dagger$ is the infinitesimal generator of the adjoint semigroup $\{S(t)^*\}$ of $\{S(t)\}$). □

Remark. We have just seen that when the pair (A,B) is stabilizable with respect to H, it is possible to construct a constant feedback $K = -N^{-1}B^*\Pi$ in $\mathcal{L}(X,U)$ in order to stabilize the pair (A,B). Hence our definition 3.1 (i) is completely equivalent to the more classical one.

The generalization of the results of W.M. WONHAM [1] to the infinite dimensional case is due to J. ZABCZYK [1].

Theorem 4.3. Let the pair (A,B) be stabilizable with respect to H and let the pair (A,H) be detectable. Then the closed loop system (4.3) is L^2-stable and Π is the unique positive self-adjoint solution of equation (4.2). □

Definition 4.4. (i) The pair (A,B) is stabilizable with respect to H if there exists a feedback K in $\mathcal{L}(X,U)$ such that the closed loop system

$$(4.4) \qquad \dot{x} = (A-BK)x, \quad x(0) = x_0$$

be L^2-stable with respect to H and with respect to K.

(ii) The pair (A,B) is stabilizable if there exists a feedback K in $\mathcal{L}(X,U)$ such that the closed loop system (4.4) be L^2-stable. □

5. Relationships between controllability and stabilizability

Assume for a moment that $X = \mathbb{R}^n$, $U = \mathbb{R}^m$, $Y = \mathbb{R}^r$ and that the operators A and B are matrices of dimensions n×n and n×m, respectively. This is the so-called finite dimensional case, where the concept of controllability is defined and characterized as follows:

Definition 5.1. The pair (A,B) is said to be controllable if

$$(5.1) \qquad \forall x_0 \in X, \exists T > 0, \exists v \in L^2(0,T;U) \text{ such that } x(T;x_0,v) = 0,$$

where $x(t;x_0,v)$ is the solution of the differential equation

$$(5.2) \qquad \begin{cases} \dot{x}(t) = Ax(t) + Bv(t), & t \geq 0 \\ x(0) = x_0. \end{cases} \quad \square$$

Theorem 5.2. The following conditions are equivalent:

(i) (A,B) controllable;

(ii) Given any spectrum σ of a real n×n matrix, there exists an m×n matrix K such that the spectrum, $\sigma(A+BK)$, of A+BK is exactly σ;

(iii) Rank $[A,AB,\ldots,A^{n-1}B] = n$. □

This theorem now says that when the pair (A,B) is controllable it is necessarily stabilizable by feedback. The converse is obviously not true.

When X, U and Y are infinite dimensional spaces. Definition 5.1 can be retained, but conditions (ii) and (iii) are difficult to generalize. However the following straightforward result remains true.

Theorem 5.3. The pair (A,B) is stabilizable if the pair (A,B) is controllable. □

For the infinite dimensional problem, the concept of controllability (to the origin) as introduced in Definition 5.1 differs from the concepts of exact or approximate reachability as studied by H.O. FATTORINI [1] to [3], D.L. RUSSELL [1] to [5], R. TRIGGIANI [1] to [6] and M. SLEMROD [1] to [4]. The last two authors have done an extensive study of the relationship between the two concepts of reachability and stabilizability.

6. Examples

We shall give in this section a few examples for which the general theory developped here may be applied.

6.1. Second order parabolic systems.

This type of problem is studied in full detail in the book of J.L. LIONS [1]. The reader will find in this reference numerous examples. Notice that for such systems the operator A (and hence A*) is stable. As a result the conditions of stabilizability and detectability are automatically verified.

6.2. First order hyperbolic systems.

Such problems have been studied by N. BARDOS [1] and J.L. LIONS [1]. Under appropriate hypotheses we can make sense of such problems for distributed controls.

6.3. Boundary control.

In many distributed parameter systems, the concontrol is exerted on the boundary and the previous framework is not completely appropriate for that situation. Boundary control problems have been studied by D.L. RUSSELL [1] to [5], GRAHAM-RUSSELL [1], R. DELVER [1] and H.O. FATTORINI [1].

In some instances it is possible to reformulate the problem in such a way that the boundary control becomes a distributed control. This can be achieved by lifting the original problem to a big enough space that the control become distributed (cf. V.P. KHATSKEVICH [1], B. FRIEDMAN [1]).

Another approach has been suggested by A.V. BALAKRISNAN [2]; it results in a distributed control problem with respect to the derivative $\dot{v}$ and the initial value v(0) of the original boundary control v. Although our theory does not apply directly to these systems, similar methods can be developped (see A.V. BALAKRISNAN [2]).

An interesting example of boundary control is given by hereditary differential systems (cf. DELFOUR-MITTER [1]). When we consider the system in state form this system is controlled through a differential equation on the boundary. This problem cannot be dealt with in the present framework if we choose as state space the continuous function or a Sobolev space. However in the product space $M_2 = \mathbb{R}^n \times L^2(-a,0;\mathbb{R}^n)$ this problem reduces to a distributed control problem and the above theory can be readily applied (cf. DELFOUR-McCALLA-MITTER [1]). Detailed result on exact and approximate reachability and the relationship with stabilizability are now available (cf. MANITIUS-TRIGGIANI [1] to [3]).

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