STOCHASTIC QUANTIZATION[1]

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1. INTRODUCTION

In recent work [1] we have studied stochastic differential equations related to the free field and $(\varphi^4)_2$-fields in finite volume following the earlier work of Jona-Lasinio and Mitter [2]. In [3] we have studied Lattice approximations to these stochastic differential equations and proved a limit theorem when the lattice spacing goes to zero. We now describe the nature of the results we have obtained.

Let $\Lambda \subset \mathbf{R}^2$ be a finite open rectangle and S' denote $\mathcal{D}'(\Lambda)$ the space of distributions on Λ and let $\mathcal{S}'$ denote the space of tempered distributions on Λ. Let $C_i = (-\Delta+I)^{-1}$, i = 1,2 with Dirichlet (resp. free) boundary conditions on Λ. C_i, i = 1,2 are covariance operators and for C a covariance operator let $C(\cdot,\cdot)$ denote its integral kernel, C^α its α^{th} operator power and let μ_C denote the centered Gaussian measure with variance operator C. Consider the following S'-valued stochastic differential equation

$$\begin{cases} d\varphi(t) = -\dfrac{1}{2} C_1^{-\varepsilon} \varphi(t)dt + dw(t) \\ \varphi(0) = \phi \in S', \ 0 < \varepsilon < 1 \end{cases} \tag{1.1}$$

where W(t) is a Wiener process with covariance $C_1^{1-\varepsilon}$. It is not difficult to prove that this equation has a unique solution and has a path continuous version as an $H^{-\alpha}$-valued process on $(0,\infty)$. Moreover $\varphi(\cdot)$ is ergodic and has μ_{C_1} as it unique invariant measure. The same claims can be made with C_1 replaced by C_2. This procedure of creating a stochastic differential equation with unique invariant measure a desired invariant measure is termed stochastic quantization. It is worth observing that the random field $\varphi(t)$ for each t is a Markov random field and satisfies the Osterwalder-Schrader axioms. A proof of this will follow from that of Nelson [4]. Note that we cannot take ε=0 in equation (2.1), since the transition probabilities $p(t; \varphi,.)$ of the process φ for different t's are no longer mutually absolutely continuous, a fact needed to prove ergodicity of the process $\varphi(\cdot)$. The case ε=1 is excluded since W(t) is then no longer a genuine Wiener process.

Since the process $\varphi(\cdot)$ is ergodic with unique invariant measure μ_{C_1}, correlation functions

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$E_{\mu_{C_1}}(\tilde{\varphi}(x_1)\ldots\tilde{\varphi}(x_n))$, (($\tilde{\varphi}$) denotes the gaussian random field with covariance μ_{C_1}) can be computed by exchanging time and space averages. This is the basic idea behind Monte Carlo calculations of statistics of the random field.

We study this differential equation in a space of distributions since the invariant measure μ_{C_1} can only be supported in some space of distributions. This is a consequence of the Minlos Theorem. It can be shown that the measure μ_{C_1} is supported in the space $H^{-1}(\Lambda)$, the dual of the Sobolev space $H^1(\Lambda)$.

In [1] and [3], we have also studied the infinite-dimensional non-linear stochastic differential equation

$$d\varphi(t) = -\frac{1}{2}(C^{-\varepsilon}\varphi(t) + C^{1-\varepsilon}{:}\,\varphi(t)^3{:})dt + dw(t) \tag{1.2}$$

with $\varphi(0)$ having initial law μ given by:

$$\left.\begin{aligned} \frac{d\mu}{d\mu_{C_1}} &= \exp\left(-\frac{1}{4}\int_\Lambda {:}\phi^4{:}dx\right)\Big/Z \\ Z &= \int_\Lambda \exp\left(-\frac{1}{4}\int {:}\varphi^4{:}dx\right) d\mu_{C_1}(\varphi) \end{aligned}\right\} \tag{1.3}$$

In the above $:\varphi(t)^3:$ denotes Wick-ordering with respect to μ_{C_1} and has the explicit definition:

$${:}\varphi(t)^3{:} = \varphi^3(t) - 3\left(E_{\mu_{C_1}}\varphi(t)^2\right)\varphi(t) \tag{1.4}$$

and is well-defined as an element of $L^2(d\mu_{C_1})$. Similarly $:\varphi^4:$ denotes WIck-ordering with respect to μ_{C_1} and the integral $\int_\Lambda {:}\varphi^4{:}dx$ is well-defined as an element of $L^2(d\mu_C)$ via an appropriate limiting procedure. The fact that μ is a well-defined probabability measure is a consequence of Nelson's estimate [4].

The difficulty of studying equation (1.2) is that since the non-linear drift term $:\varphi(t)^3:$ is only defined in some limiting sense we cannot interpret it in the Ito snese and hence we have to interpret it in a weak sense. In [1] it is shown that the new measure P_0 defined by

$$\frac{dP_0}{dP} = \exp\Big(\frac{1}{2}\int_0^T <{:}\varphi^3(s){:}, dw(s)> - \frac{1}{8}\int_0^T <{:}\varphi^3(s){:}, C^{1-\varepsilon}{:}\,\varphi^3(s){:}>_{\dot{H}^{\alpha}, H^{\alpha}} ds$$

$$+ \frac{1}{4}\int_\Lambda {:}\varphi^4(0){:}\, dx\Big)\Big/Z \tag{1.5}$$

where Z is a normalizing constant, is a well-defined probability measure. The proof uses both estimates from quantum field theory and probabilistic arguments (in particular Novikov's criterion for an exponential super-martingale to be a martingale).

In [1] a limit theorem at the process level when $\Lambda \uparrow \mathbf{R}^2$ is also proved.

2. STOCHASTIC QUANTIZATION AND IMAGE ANALYSIS

Our interest in these problems arose from problems of image analysis. To see this note that the measure μ corresponds to Hamiltonian

$$H = \int_{\Lambda} \left[\|\nabla\varphi\|^2 + m_0^2\,\varphi^2 + \lambda{:}\varphi^4{:} \right] dx \tag{2.1}$$

where m_0 is the bare mass and λ the coupling constant (taken both to be 1 in the previous section). Corresponding to the Hamiltonian we can construct the limit Gibbs measure in the sense of Sinai (cf. [5] and [4]).

Consider the following problems of Image Analysis.

Problem I.

Let $\Omega \subset \mathbf{R}^2$ be an open bounded set and let $\psi \in L^\infty(\Omega)$ be given. We think of ψ as an observed noisy image. We wish to construct an estimate $\varphi \in H^1(\Omega)$ such that

$$J(\varphi) = \int_{\Omega} |\psi - \varphi|^2 dx + \int_{\Omega} \|\nabla\varphi\|^2 dx$$

is minimized.

It is natural to think of $J(\varphi)$ as a conditional Hamiltonian $H_0(\varphi|\psi)$ and construct a conditional measure $\mu(\varphi|\psi)$ by making appropriate probabilistic hypotheses on ψ (for example by associating an Hamiltonian for ψ). To construct estimates we would have to compute statistics corresponding to the measure $\mu(\varphi|\psi)$ and this would be done using the ideas of stochastic quantization for both φ and ψ. A start towards doing this has been made in [6].

Problem II.

Let $\Omega \subset \mathbf{R}^2$, be bounded and open and let $\psi \in L^\infty(\Omega)$. Consider the following variational problem. Minimize

$$J(\varphi, \Gamma) = \int_{\Omega} |\psi - \varphi|^2 dx + \int_{\Omega\setminus\Gamma} |\nabla\varphi|^2 dx + \mathcal{H}^1(\Gamma),$$

where Γ is a closed set with $\Gamma \subset \bar{\Omega}$ and $\mathcal{H}^1(\Gamma)$ denotes the one-dimensional Hausdorff measure. The interpretation of this functional is that we want to find an estimate $(\hat{\varphi}, \hat{\Gamma})$ of the observed noisy image ψ which preserves the discontinuities of the image, there are not too many discontinuities and $\hat{\Gamma}$ is an estimate of the discontinuities. It can be shown that a minimizing solution $(\hat{\varphi}, \hat{\Gamma})$ exists [7], [8]. A detailed study of the first variation of J has been done in [9].

It is not clear how to give a probabilistic interpretation to this problem. However, if we consider a lattice analog, then we can give a probabilistic interpretation by constructing a measure on the lattice $Z^2x(Z^2)^*$, where $(Z^2)^*$ denotes the dual lattice. This was one of the motivations for our work reported in [3]. For details of this problem in a discrete space setting, see our paper [10] and the references cited there.

3. RENORMALIZATION GROUP METHODS AND A BELLMAN EQUATION

The main purpose of this section is to describe the renormalization group method of K.G. Wilson for U-V cut-off removal as formulated by P.K. Mitter [11, 12]. A certain infinite-dimensional Hamilton-Jacobi-Bellman equation arises in this context which has a natural control-theoretic interpretation.

Consider the linear parabolic equation in $\mathbf{R}^n$ x(0,T]

$$\left.\begin{aligned} dp^\varepsilon(x,t) &= L_\varepsilon^* p^\varepsilon(x,t) + \frac{1}{\varepsilon} V(x,t)p^\varepsilon(x,t) \\ p^\varepsilon(x,0) &= p_0^\varepsilon(x) = K_\varepsilon \exp\left(-\frac{1}{\varepsilon} S_0(x)\right) \end{aligned}\right\} \tag{3.1}$$

Here $\varepsilon > 0$, $S_0(x) > 0$, $\lim\limits_{\varepsilon\to 0} \varepsilon \ell n\, K_\varepsilon = 0$ and L_ε^* is the formal adjoint of the diffusion operator

$$L_\varepsilon = \frac{\varepsilon}{2} \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{n} f_i(x) \frac{\partial}{\partial x_i} \tag{3.2}$$

We assume that f is a C^∞-function with bounded derivatives upto order 3, -V is a C^∞-function which is bounded below by zero.

Following, for example, Fleming-Mitter [13], introduce the logaarithmic transformation

$$S^\varepsilon(x,t) = -\varepsilon \ln p^\varepsilon(x,t). \tag{3.3}$$

Then $S^\varepsilon(x,t)$ satisfies the Bellman-Hamilton-Jacobi equation

$$\left.\begin{aligned} &\frac{\partial}{\partial t} S^\varepsilon(x,t) - \frac{\varepsilon}{2} \Delta S^\varepsilon(x,t) + H^\varepsilon(x,t,\nabla S(x,t)) = 0 \\ &S^\varepsilon(x,0) = -\varepsilon \ln p_0^\varepsilon(x), \end{aligned}\right\} \tag{3.4}$$

and $H^\varepsilon(x,t,p) = p'f(x) + \frac{1}{2}\|p\|^2 - V(x,t)$.

Formally, letting $\varepsilon \to 0$, we obtain the Hamilton-Jacobi equation

$$\left.\begin{aligned} &\frac{\partial}{\partial t} \tilde{S}(x,t) + H(x,t,\nabla\tilde{S}(x,t)) = 0, \\ &S(x,0) = S_0(x) . \end{aligned}\right\} \tag{3.5}$$

One can prove that $\lim_{\varepsilon\to 0} \varepsilon \ln p^{\varepsilon}(x,t) = -J(x,t)$ on compact subsets of $\mathbf{R}^n \times [0,T]$, where $J(x,t)$ is the value function of a deterministic optimal control problem:

Minimize

$$J(t;\, x_0,\, u) = S_0(x_0) + \frac{1}{2}\int_0^t \|u(s)\|^2 ds \tag{3.6}$$

subject to

$$\left.\begin{aligned} \frac{dx}{ds} &= f(x(s)) + u(s) \\ x(0) &= x_0. \end{aligned}\right\} \tag{3.7}$$

Let $U_{x,t} = \{(x_0, u) | x_u(0) = x_0,\, x_u(t) = x,\, u \in L^2(0,t;\mathbf{R}^n)\}$, and

$$J(x,t) = \mathrm{Inf}[J(t;\, x_0,u) | (x_0,\, u) \in U_{x,t}].$$

Then finally J satisfies (3.5). Note that this is a minimum energy optimum control problem. In a similar manner, $S^{\varepsilon}(x,t)$ has the interpretation of a value function for a Markovian stochastic opitmal control problem [12].

We now return to the ideas of section 1. We consider the random field $\phi(x)$ on $\mathbf{R}^d$, $d>2$ with measure μ_C. The covariance C has a kernel $C(x-y)$ given by the formula (in terms of Fourier transforms)

$$C(x-y) = \frac{1}{(2\pi)^d}\int d^d\omega\, \frac{1}{\omega^2}\, e^{i\omega.(x-y)}$$

(the covariance operator is $(-\Delta)^{-1}$ in contrast to the covariance operator $(-\Delta+I)^{-1}$ in Section 1). Let the measure μ_{C_κ} be defined by giving the kernel

$$C_\kappa(x-y) = \int \frac{d^d\omega}{(2\pi)^d}\, \frac{e^{-\frac{\omega^2}{\kappa^2}}}{\omega^2}\, e^{i\omega.(x-y)}$$

A computation gives the scaling properties

$$C_\kappa(x-y) = \kappa^{d-2} C_1(\kappa(x-y)), \tag{3.8}$$

and if ϕ denotes the random field with measure μ_{C_κ} given by covariance C_κ and Φ denotes the random field with measure μ_{C_1} given by covariance C_1, then

$$\phi(x) = \kappa^{\frac{d-2}{2}}\, \Phi(\kappa x). \tag{3.9}$$

The measure μ_{C_κ} is supported on smooth functions. By virtue of the above

$$E_{\mu_{C_\kappa}}(\phi(x_1)\ldots\phi(x_n)) = \kappa^{n(\frac{d-2}{2})} E_{\mu_{C_1}}(\Phi(\kappa x_1)\ldots\Phi(\kappa x_n)) \tag{3.10}$$

The problem of studying the behaviour of the n-point correlations for fixed $x_1,\ldots,x_n$ as $\kappa \to \infty$ is equivalent to studying the long distance (infinite volume limit) problem at a fixed cut-off.

Let $V_0(\phi)$ be an even polynomial and consider the new measure with interaction V_0

$$d\mu_\kappa = d\mu_{C_\kappa} \exp(-V_0(\phi)) \tag{3.11}$$

and the corresponding characteristic function

$$Z_\kappa(f) = \int d\mu_\kappa \exp(i\phi(f)) \tag{3.12}$$

There are two steps in the renormalization group method.

Step 1 (Scaling)

From (3.9),

$$d\mu_\kappa(\Phi) = d\mu_{C_1}(\Phi) \exp\left(-V_0\left(\kappa^{\frac{d-2}{2}} \Phi(\cdot)\right)\right). \tag{3.13}$$

Set

$$V_0\left(\kappa^{\frac{d-2}{2}} \Phi(\cdot) = \mathcal{V}_0^{(\kappa)}(\Phi(\cdot)\right).$$

Then

$$Z_\kappa(f) = \int d\mu_{C_1}(\Phi)\exp\left(-\mathcal{V}_0^{(\kappa)}(\Phi(\cdot)) + \Phi(f_\kappa)\right) \tag{3.14}$$

where $f_\kappa(x) = \kappa^{\frac{d-2}{2}-d} f(\kappa^{-1}x)$.

Step 2. Lowering the Cut-Off.

Consider the transformation

$$1 \to e^{-t}.1, \quad t \in R_+ .$$

We know,

$$C_1(x-y) = \int \frac{d^d\omega}{(2\pi)^d} \frac{e^{(-\frac{\omega^2}{1^2})}}{\omega^2} e^{i\omega.(x-y)},$$ and hence

$$C_{e^{-t}.1}(x-y) = \int \frac{d^d\omega}{(2\pi)^d} \frac{e^{\left(\frac{-\omega^2}{e^{-2t}}\right)}}{\omega^2} e^{i\omega.(x-y)} \tag{3.15}$$

Now $C_1 > C_{e^{-t}.1}$ as operators.

Let $C_1 = C_{e^{-t}.1} + C_t^{(h)}$. (3.16)

In the above C_1 is the covariance of the field Φ at unit cut-off, $C_{e^{-t}.1}$ the covariance corresponding to the lowered cut-off and $C_t^{(h)}$ the covariance corresponding to a fluctuating field.

From the (3.16) we have the decomposition $\Phi = \phi^{(1)} + \zeta$, ζ denoting the fluctuating field and $\phi^{(1)}$ and ζ are independent Gaussian field. The covariance kernel of $C_t^{(h)}$ has exponential decay as $|x-y| \to \infty$.

We now integrate out the fluctuating field and scale back.

$$\int d\mu_{C_1}(\Phi) \exp(-\mathcal{V}_0^{(\kappa)}(\Phi)) = \int d\mu_{C_{e^{-t}.1}}(\phi^{(1)}) d\mu_{C_t}(h) \exp(-\mathcal{V}_0^{(\kappa)}(\phi_1+\zeta))$$

$$= \int d\mu_{C_1}(\Phi) d\mu_{C_t^{(h)}}(\zeta) \exp\left[-\mathcal{V}_0^{(\kappa)}\left(e^{-\frac{d-2}{2}t}\Phi(e^{-t}.) + \zeta\right)\right].$$

The renormalization group transformation is defined by

$$\exp(-\mathcal{V}_t^{(\kappa)}(\Phi)) = \int d\mu_{C_t^{(h)}}(\zeta) \exp\left[-\mathcal{V}_0^{(\kappa)}\left(e^{-\frac{d-2}{2}t}\Phi(e^{-t}.) + \zeta\right)\right] \tag{3.17}$$

which sends

$$\mathcal{V}_0^{(\kappa)} \to \mathcal{V}_t^{(\kappa)}.$$

$\mathcal{V}_t^{(\kappa)}$ is called the effective potential.

A computation shows that $\mathcal{V}_t$ (dropping the superscript κ) satisfies the infinite-dimensional Bellman-Hamilton-Jacobi equation

$$\frac{\partial \mathcal{V}_t}{\partial t} = -\int d^d x \left(\left[\frac{d-2}{2} + x.\nabla_x\right]\Phi(x)\right) \frac{\delta \mathcal{V}_t}{\delta\Phi(x)}$$

$$-\int d^d x \,.\, d^d y\, K(x-y)\left[-\frac{\delta^2 \mathcal{V}_t}{\delta\Phi(x)\delta\Phi(y)} + \frac{\delta \mathcal{V}_t}{\delta\Phi(x)} \frac{\delta \mathcal{V}_t}{\delta\Phi(y)}\right] \tag{3.18}$$

where

$$K(x-y) = \int \frac{d^d\omega}{(2\pi)^d} e^{-i\omega.(x-y)} e^{-\omega^2}.$$

$\nu_0^{(K)}$ will have parameters which will have to be fixed so that we start at a critical surface. Studying the fixed point of the renormalization group transformation is equivalent to studying the asymptotic behavior of the equation (3.18) (at least in the small region).

Equation (3.18) has a stochastic control interpretation as suggested earlier in the section, and $\nu_t^{(K)}$ has the interpretation of a Bellman Value function. The machinery of non-linear semigroups may be useful for this purpose.

4. NEW PROBLEMS

We would like to suggest that the ideas of the renormalization group method as exposed in the previous section could be generalized to yield a dynamic renormalization group method which would be relevant to problems of stochastic quantization. A program for this is described below.

We consider the stochastic differential equation (1.1). The solution of this equation for each t gives us a Gaussian measure in path space. This path space Gaussian measure plays the role of the measure μ_C of section 3. Cut-offs can be introduced for this measure and scaling properties analogous to (3.8) and (3.9) obtained. Note that this Gaussian measure can be obtained via a Girsanov Transformation of Wiener measure. The interaction measure is now introduced by a second Girsanov transformation as in (1.5). The proposal is to proceed as in Section 3 where the renormalization group transformation is now a transformation of Girsanov functionals thereby creating an effective Girsanov functional. The details of this will be presented elsewhere.

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