LATTICE APPROXIMATION IN THE STOCHASTIC QUANTIZATION OF $(\phi^4)_2$ FIELDS[1]

Vivek S. Borkar
Tata Institute for Fundamental Research (TIFR)
P. O. Box 1234, Bangalore, India

Sanjoy K. Mitter
Department of Electrical Engineering and Computer Science
Laboratory for Information and Decision Systems (LIDS)
Center for Intelligent Control Systems
Massachusetts Institute of Technology
Cambridge, MA 02139
U.S.A.

I. INTRODUCTION

The Parisi-Wu program of stochastic quantization [8] involves construction of a stochastic process which has a prescribed Euclidean quantum field measure as its invariant measure. This program was rigorously carried out for a finite volume $(\phi^4)_2$ measure by G. Jona-Lasinio and P. K. Mitter in [6]. These results were extended in [2], which also proves a finite to infinite volume limit theorem. The aim of this note is to prove a related limit theorem, viz., that of the finite dimensional processes obtained by stochastic quantization of the lattice $(\phi^4)_2$ fields to their continuum limit, i.e., the $(\phi^4)_2$ process of [2], [6]. The proof imitates that of the limit theorem of [2] in broad terms, though the technical details differ. Note that this limit theorem can also be construed as an alternative construction of the $(\phi^4)_2$ process in finite volume.

The next section recalls the finite volume $(\phi^4)_2$ process. Section III summarizes the relevant facts about the lattice approximation to the $(\phi^4)_2$ field from Sections 9.5 and 9.6 of [4]. Section IV proves the limit theorem.

[1]The research of the second author was supported in part by the U.S. Army Research Office, Contract NO. DAAL03-86-K-0171 (Center for Intelligent Control Systems, Massachusetts Institute of Technology), and by the Air Force Office of Scientific Research, Contract No. AFOSR-85-0227.

II. THE $(\phi^4)_2$ PROCESS

Let $\Lambda \subset R^2$ be a finite rectangle which, for simplicity, we take to be the unit cube $x = (x_1, x_2) | 0 \leq x_1, x_2 \leq 1$ Let Δ denote the Dirichlet Laplace operator on Λ. It is diagonalized by the basis $e_k(x) = 2 \sin(k_1 x_1) \sin(k_2 x_2)$, $x = (x_1, x_2)$, $k \varepsilon B = \{(k_1, k_2) | k_i = n\pi, n \geq 1, i = 1, 2\}$. In fact, $-\Delta e_k = k^2 e_k$ where $k_2 = k_1^2 + k_2^2$. For $\alpha \varepsilon R$, let H^α denote the Hilbert space obtained by completing $D(\Lambda)$ with respect to the inner product

$$< f, g >_\alpha = \sum_{k \varepsilon B} (k^2)^\alpha <f, e_k> <g, e_k>$$

where $<\cdot,\cdot>$ is the L_2 scalar product. Topologize $Q = \bigcup H^\alpha$ by the countable family of seminorms $\|\cdot\|_n = <\cdot,\cdot>^{\frac{1}{2}}$ and $Q' = \bigcup_\alpha H^{-\alpha}$ via duality.

Let $C = (-\Delta + 1)^{-1}$, $C(\cdot,\cdot)$ its integral kernel, C^α its α-th operator power, and μ_C the centered Gaussian measure on H^{-1} with covariance C [2], [6]. Let $::$ denote the Wick ordering with respect to C (see [4], Ch. 8, for a definition). The $(\phi^4)_2$ measure on H^{-1} is defined by

$$\frac{d\mu}{d\mu_C} = \exp\left(-\frac{1}{4} \int :\phi: dx\right) / Z \qquad [2.1]$$

where

$$Z = \int \exp\left(-\frac{1}{4} \int_\Lambda :\phi^4: dx\right) d\mu_C < \infty .$$

See [4], Section 8.6, for details.

Let $0 < \varepsilon < 1$ and $\beta_k(\cdot)$, $k \varepsilon B$, a collection of independent standard Brownian motions. Define

$$W(t) = \sum_{k \varepsilon B} (k^2)^{-(1-\varepsilon)/2} \beta_k(t) e_k(\cdot), \ t \geq 0 .$$

This defines an H^{-1}-valued Wiener process with covariance $C^{1-\varepsilon}$ [2], [6]. The equation

$$d\phi(t) = -\frac{1}{2} (C^{-\varepsilon} \phi(t) + C^{1-\varepsilon} :\phi^3(t):) dt + dW(t) \qquad [2.2]$$

with initial law μ can be shown to have a unique stationary weak solution as an H^{-1}-valued process, defining an ergodic process called the $(\phi^4)_2$ process. See [2], [6] for details.

^{1}, $n \geq 1\}$ and pick $\delta \varepsilon A$. The finite lattice Λ_δ with spacing follows: Let $\delta Z^2 = \{\delta z | z \varepsilon Z^2\}$, int $\Lambda_\delta =$ int $\Lambda \cap \delta Z^2$, $=$ int $\Lambda_\delta \cup \partial\Lambda_\delta = \Lambda \cap \delta Z^2$. ℓ_2(int Λ_δ) is the Hilbert space with

$$= \sum_{x \varepsilon \text{int } \Lambda_\delta} \delta^2 |f(x)|^2 \,,$$
δ

space of $\ell_2(\Lambda_\delta)$. On $\ell_2(\delta Z^2)$, define the forward gradient on α by $(\partial_{\delta,\alpha} f)(x) = \delta^{-1}[f(x+d\mu_\alpha) - f(x)]$ where μ_α is the the α-th direction for $\alpha = 1,2$. The backward gradient oint with respect to the $\ell_2(\delta Z^2)$ inner product.

$,1 + \partial^*_{\delta,2}\ \partial_{\delta,2}$. Then $(\bar\Delta_\delta f)(x) = \delta^{-2}(-4f(x) + \sum f(y))$ tion is over the nearest neighbours of x. Let Π be the $^2) \to \ell_2$(int Λ_δ). The Dirichlet difference Laplacian Δ_δ $\bar\Delta_\delta \Pi$ and agrees with $\bar\Delta_\delta$ on int Λ_δ.

a basis on ℓ_2(int Λ_δ) the $(\delta^{-1}-1)^2$ functions : int Λ_δ, $k_\alpha = \pi, 2\pi, \ldots, (\delta^{-1}-1)\pi$; $\alpha = 1,2\}$.

p. 221) $\{e_k^\alpha\}$ diagonalize $-\Delta_\delta$ with e_k^δ, $\lambda_k^\delta = 4\delta^{-2} \sum_{i=1}^{2} \sin^2(\frac{\delta k_i}{2})$.

$\text{t } \Lambda_\delta$ $= 1$ if $k = \ell$, $= 0$ otherwise

p. 222) The map $i_\delta : e_k^\delta \to e_k$ defines an isometric imbed- $_\delta) \to L_2(\Lambda)$.

the projection operator on $L_2(\Lambda)$ which truncates the t $k_\alpha/\pi = \delta^{-1}$, so that $= \sum^\delta \alpha_k e_k$ where $\sum^\delta$ denotes the summation over $\leq \pi^{-1} k_i \leq \delta^{-1}-1$, $i=1,2\}$. Then $i^*_\delta f = \Pi_\delta f |_{\Lambda_\delta}$. We can

$_\delta + 1)^{-1} : \ell_2(\text{int } \Lambda_\delta) \to \ell_2(\text{int } \Lambda_\delta)$ as an operator on $L_2(\Lambda)$, ometry, i.e., let $C_\delta = i_\delta C_\delta i^*_\delta$ where the C_δ on the right s on $\ell_2(\text{int } \Lambda_\delta)$(resp. $L^2(\Lambda)$). As an operator on $L_2(\Lambda)$, its kernel $\sum^\delta (\lambda_k^\delta + 1)^{-1} e_k(x) e_k(y)$, icted to the lattice points in int Λ_δ, coincides with es of C_δ as an operator on $\ell_2(\text{int } \Lambda_\delta)$.

pp. 222-224) $\| C_\delta - C \| \leq (0\ \delta^2)$ as operators on $L_2(\Lambda)$, $C_\delta(x,\cdot)\|_{L_p(\Lambda)} \leq O(\delta^\alpha)$ for $\alpha < (2p^{-1}, 1)$.

If ϕ is a Gaussian field with covariance C, $\phi_\delta(x) = (i^*_\delta\phi)(x)$ for $x \varepsilon$ int Λ_δ defines a Gaussian lattice field with covariance $C_\delta = i^*_\delta C_\delta i_\delta$. The field ϕ_δ can be realized by a Gaussian measure on $L_2(R^{|\text{int}\Lambda_\delta|})$. Explicitly, letting $\prod_{x\varepsilon \text{ int } \Lambda_\delta} d\phi_\delta(x)$ denote the Lebesgue measure on $R^{|\text{int } \Lambda_\delta|}$, the above measure is given by

$$d\mu_{\delta C} = (\det C_\delta)^{-\frac{1}{2}}\ \pi^{-|\text{int } \Lambda_\delta|^{\frac{1}{2}}} \exp\left(-\frac{\delta^4}{2} \sum_{x,y\varepsilon \text{ int } \Lambda_\delta} \phi_\delta(x) C_\delta^{-1}(x,y)\, \phi_\delta(y)\right) \prod_x d\phi_\delta(x).$$

This is the lattice analog of μ_C. The lattice analog of μ can now be defined as follows: Define for $f \ \varepsilon\ \ell_2(\text{int } \Lambda_\delta)$,

$$:\phi_\delta^n:(f) = \delta^2 \sum_{x\varepsilon \text{ int } \Lambda_\delta} :\phi_\delta^n(x):_{C_\delta} f(x).$$

The lattice analog μ_δ is given by

$$d\mu_\delta = \exp\left(-\frac{1}{4} :\phi_\delta^4(x):_\delta(1)\right) d\mu_{\delta C} \Big/ \int \left(\int \exp\left(-\frac{1}{4} :\phi_\delta^4:_\delta(1)\, d\mu_{\delta C}\right)\right) \qquad [3.1]$$

For $k \ \varepsilon\ B_\delta$, let $\{\beta_k(\cdot)\}$ be a collection of independent standard Brownian motions. For $0 < \varepsilon < 1$, define

$$B_\delta(t) = \delta^2 \sum^\delta (\lambda_k^\delta + 1)^{-(1-\varepsilon)/2}\ \beta_k(t)\, e_k(\cdot), \quad t \geq 0.$$

This defines an $L_2(\Lambda)$-valued Wiener process with covariance $C_\delta^{1-\varepsilon}$. The analog of [2.2] in the lattice case is

$$d\phi_\delta(t) = \frac{1}{2}\left(C_\delta^{-\varepsilon}\ \phi_\delta(t) + C_\delta^{1-\varepsilon} : \phi_\delta^3(t):_\delta\right) dt + dB_\delta(t) \qquad [3.2]$$

where the operators act on $L_2(\Lambda)$. $\phi_\delta(\cdot)$ is viewed here as an $L_2(\Lambda)$-valued process. However, letting $\phi_\delta(t) = \sum^\delta \phi_{\delta k}(t) e_k$, [3.2] translates into an equivalent stochastic differential equation for finitely many scalar processes $\phi_{\delta k}(\cdot)$ with locally Lipschitz (in fact, polynomial) coefficients. This ensures the existence of an a.s. unique strong solution to [3.2] up to an explosion time. That it does not explode a.s. is proved by a standard application of Khasminskii's test for non-explosion exactly as in [G], Section 3.

By identifying the vector $\{\phi_\delta(x), x\varepsilon \text{ int } \Lambda_\delta\}$ with $\phi_\delta(\cdot) \varepsilon\ \ell_2(\text{int } \Lambda_\delta)$, μ_δ can be considered as a probability measure on $\ell_2(\text{int } \Lambda_\delta)$ and via the isometry i_δ, as a probability measure on $L_2(\Lambda)$. We retain the notation μ_δ for the latter interpretation, as only this interpretation will be used henceforth. A computation similar to that of [2], Section 3, shows that the generator of the Markov process described by [3.2] is self-adjoint on $L_2(\mu_\delta)$. By Theorem 2.3 of [3], the same holds for the associated transition semigroup of $\{T_t,\ t \geq 0\}$ of operators on $L_2(\mu_\delta)$. Thus for $f, g\ \varepsilon L_2(\mu_\delta)$, $\int f\, T_t g\, d\mu_\delta = \int (T_t f) g\, d\mu_\delta$. Letting $f(\cdot) \equiv 1$, $\int T_t g\, d\mu_\delta = \int g\, d\mu_\delta$, implying that μ_δ is an invariant probability measure for $\phi_\delta(\cdot)$. In fact, the resulting process will be ergodic. We won't need this fact here, so we omit the details. From now on, [3.2] will always be considered with initial law μ_δ.

IV. THE CONTINUUM LIMIT

This section establishes the main result of this paper, viz., the convergence of $\phi_\delta(\cdot)$ to the $(\phi^4)_2$ process as $\delta \to 0$ in A, in the sense of weak convergence of Q′-valued processes. Thus we consider $\phi_\delta(\cdot)$ as a Q′-valued process and μ_δ as a measure on Q′ via the injection of $L_2(\Lambda)$ into Q′. From theorem 9.6.4, p.228, [4], it follows that the finite dimensional marginals of the collection $\{\phi_\delta(e_k), k\epsilon B\}$ under μ_δ converge weakly to the corresponding ones under μ as $\delta \to 0$ in A. Since μ_δ, μ are supported on H^{-1}, it follows that $\mu_\delta \to \mu$ weakly as probability measures on Q′. (A proof of the former assertion would go as follows: Since H^{-1} is Polish, it is homeomorphic to a G_μ subset of $[0,1]^\infty$ whose closure $\bar{H}^{-1}$ can be considered a compactification of H^{-1}. As a measure on $\bar{H}^{-1}$, $\{\mu_\delta\}$ are tight and for any weak limit point υ thereof, its restriction υ' to H^{-1} must yield the same finite dimensional marginals for $\{\phi(e_k), k\epsilon B\}$ as μ. Thus $\upsilon = \upsilon' = \mu$.) As a first step towards proving the continuum limit, we prove some tightness results.

Let

$$\phi_{\delta_1}(t) = \phi_\delta(t)$$
$$\phi_{\delta_2}(t) = \frac{1}{2}\int_0^t C_\delta^{-\varepsilon}\phi_\delta(s)\,ds$$
$$\phi_{\delta_3}(t) = \frac{1}{2}\int_0^t C_\delta^{1-\varepsilon} :\phi_\delta^3(s):\,ds$$
$$\phi_{\delta_4}(t) = B_\delta(t)$$

for $t \leq 0$. Pick $t_1 \leq t_2 0$ in $[0,T]$, $\infty > T > 0$. In what follows, K denotes a positive constant (not always the same) that may depend on T, but not on δ. Let $f\epsilon Q$

<u>Lemma 4.1</u> $$E[(\int_{t_1}^{t_2} C_\delta^{-\varepsilon}\ \phi_\delta(t)(f)\,dt)^4] \leq K|t_2-t_1|^2 \qquad [4.1]$$

<u>Proof</u> Using Jensen's inequality and stationarity of $\phi_\delta(\cdot)$, one obtains

$$E[(\int_{t_1}^{t_2} C_\delta^{-\varepsilon}\ \phi_\delta(t)(f)\,dt)^4] \leq K|t_2-t_1|^2\ E[|C_\delta^{-\varepsilon}\phi_\delta(0)(f)|^4].$$

Letting $\Lambda_\delta = d\mu_\delta/d\mu_{\delta C}$, the expectation on the right is bounded by

$$[\int |C_\delta^{-\varepsilon}\phi(f)|^8\,d\mu_{\delta C}(\phi)]^{\frac{1}{2}}\ [\int \Lambda_\delta^2\,d\mu_{\delta C}]^{\frac{1}{2}}.$$

By Lemma 9.6.2, p. 227, [4], the second term above is bounded uniformly in δ. Using Feynman graph calculations, as in Theorem 8.5.3, p. 191, [4], one has

$$\int |C_\delta^{-\varepsilon}\phi(f)|^8\,d\mu_{\delta C}(\phi) \leq K\,\|C_\delta^{-\varepsilon} f\|_2^8.$$

Now

$$\|C_\delta^{-\varepsilon} f - C^{-\varepsilon} f\|_2^2 = \sum_{k\epsilon B} \langle f, e_k\rangle^2 ((\lambda_k^\delta + 1)^\varepsilon\ I\{k\epsilon B_\delta\} - (\lambda_k + 1)^\varepsilon)^2.$$

The summand on the right can be dominated in absolute value by $K\langle f, e_k\rangle^2 \lambda_k^2$ which is summable for $f\epsilon Q$. By the dominated convergence theorem,

$$\lim \|C_\delta^{-\varepsilon} f - C^{-\varepsilon} f\|_2 = 0,$$

implying $\sup_\delta \|C_\delta^{-\varepsilon} f\|_2 < \infty$. [4.1] follows. QED

<u>Lemma 4.2</u> $$E[(\int_{t_2}^{t_2} C_\delta^{1-\varepsilon} :\phi_\delta^3(t):(f)\,dt)^4] \leq K|t_2-t_1|^2. \qquad [4.2]$$

This follows along similar lines.

<u>Lemma 4.3</u> $$E[(|B_\delta(t_2)(f) - B_\delta(t_1)(f)|^4] \leq K|t_2-t_1|^2. \qquad [4.3]$$

<u>Proof</u> The lefthand side equals

$$3|C_\delta^{-\varepsilon}(f,f)|^2\,|t_2-t_1|^2 \leq 3\sup_\delta \|C_\delta^{(1-\varepsilon)/2} f\|_2^2\,|t_2, t_1|^2.$$

As in the proof of Lemma 4.1, one can prove

$$\lim_{\delta\to 0} \|C_\delta^{(1-\varepsilon)/2} f - C^{(1-\varepsilon)/2} f\|_2 = 0.$$

Thus $\sup_\delta \|C_\delta^{(1-\varepsilon)} f\|_2 < \infty$ and the claim follows. QED

<u>Corollary 4.1</u> $$E[|\phi(t_2)(f) - \phi(t_1)(f)|^4 \leq K|t_2-t_1|^2 \qquad [4.4]$$

<u>Proof</u> Follows from [3.2] and [4.1] - [4.3]. QED

<u>Lemma 4.4</u> The laws of the processes $[\phi_{\delta_1}(\cdot), \phi_{\delta_2}(\cdot), \phi_{\delta_3}(\cdot), \phi_{\delta_4}(\cdot)]$ viewed as $(C(0,\infty); Q'))^4$-valued random variables remain tight as δ varies over A.

<u>Proof</u> By Theorem 3.1 of [7], it suffices to establish the tightness of $[\phi_{\delta_1}(\cdot)(f), \phi_{\delta_2}(\cdot)(f), \phi_{\delta_3}(\cdot)(f), \phi_{\delta_4}(\cdot)(f)]$ on $[0,T]$ as $(C([0,T]; R))^4$-valued random variables for arbitrary $T > 0$ and $f\epsilon Q$. This, however, is immediate from the tightness of $\{\mu_\delta\}$ (since $\mu_\delta \to \mu$ weakly as a measure on H^{-1}), the estimates [4.1] - [4.4] and the criterion of [1], p. 95. QED

Recall that a family of probability measures on a product of Polish spaces is tight if and only if its images under projection onto each factor space are. Letting $\{\bar{e}_i\}$ denote an enumeration of $\{e_k\}$. This implies, in view of the foregoing, that $[\phi_{\delta_1}(\cdot)(\bar{e}_1), \ldots, \phi_{\delta_4}(\cdot)(\bar{e}_1), \phi_{\delta_1}(\cdot)(\bar{e}_2), \ldots, \phi_{\delta_4}(\cdot)(\bar{e}_2), \phi_{\delta_1}(\cdot)(\bar{e}_3), \ldots]$ are tight as $(C([0,\infty]; R))^\infty$-valued random variables. By dropping to a subsequence of A, denoted by A again, we may assume that they converge in law as $\delta \to 0$ along A. Then for any finite subset $\{t_1, \ldots, t_k\}$ of $[0,\infty]$ and a collection $\{g_1, \ldots, g_k\}$ of finite linear combinations of $\{\bar{e}_i\}$, the ıws of $\{\phi_{\delta 1}(t_j)(g_j),\ 1 \le i \le 4, 1 \le j \le k\}$ converge. Consider a .on $f_1, \ldots, f_k$ in Q. Using the kind of estimates used in the ›f Lemmas 4.1 - 4.3, we have

$$:|\phi_{\delta 1}(t_j)(f_j - g_j)|^2] \le M\|f_j - g_j\|_2^2 \qquad [4.5]$$

$$:|\phi_{\delta 2}(t_j)(f_j - g_j)|^2] \le M\|C_\delta^{-\varepsilon}(f_j - g_j)\|_2^2 \qquad [4.6]$$

$$:|\phi_{\delta 3}(t_j)(f_j - g_j)|^2] \le M\|C_\delta^{1-\varepsilon}(f_j - g_j)\|_2^2 \qquad [4.7]$$

$$:|\phi_{\delta 4}(t_j)(f_j - g_j)|^2] \le M\|C_\delta^{(1-\varepsilon)/2}(f_j - g_j)\|_2^2 \qquad [4.8]$$

ıitable constant M depending on max $(t_1, \ldots, t_k)$. As $\delta \to 0$ in A, ıthand sides of [4.6] - [4.8] converge to the corresponding quan- ›ith C replacing C_δ. Since g_j can be obtained by suitably trun- :he Fourier series of f_j in $\{e_i\}$, each of these limiting expres- ıd the righthand side of [4.5] can be made smaller than any pre- $\eta > 0$ uniformly in $1 \le j \le k$ by a suitable choice of $\{g_j\}$. It that the righthand sides of [4.5] - [4.8] can be made smaller ʼ prescribed $\eta \to 0$ uniformly in $\delta \varepsilon A$ and $1 \le j \le k$ by a suitable ›f $\{g_j\}$.

ıt $\{h_\ell\}$ be an enumeration of finite linear combinations of $\{\bar{e}_i\}$:ional coefficients. By a well-known theorem of Skorohod ([5], ›e can construct on some probability space random variables $_{ij\ell}$, $\delta\varepsilon A$, $1 \le i \le 4$, $1 \le j \le k$, $\ell \ge 1$, such that $\{X_{\delta ij\ell}\}$ agrees in ı $\{\phi_{\delta i}(t_j)(h_\ell)\}$ for each fixed δ and $X_{\delta ij\ell} \to Y_{ij\ell}$ a.s. as $\delta \to 0$ 3y augmenting this probability space, if necessary, we may con- ›n it random variables $Z_{\delta ij}$, (δ, i, j) as above, such that the ıw of $[\phi_{\delta i}(t_j)(f_j), \phi_{\delta i}(t_j)(h_1), \phi_{\delta i}(t_j)(h_2), \ldots]$ agrees with that $_j$, $X_{\delta ij1}$, $X_{\delta ij2}$, ...] for each δ, i, j. Since $X_{\delta ij\ell} \to Y_{ij\ell}$ a.s. $_{\delta ij\ell}|^4] = E[|\phi_{\delta i}(t_j)(h_\ell)|^4]$ can be bounded uniformly in δ for each ʼ estimates analogous to [4.5] - [4.8], we have $E[|X_{\delta ij\ell} - Y_{ij\ell}|^2] \to 0$ in A for each i, j, ℓ. On the other hand, given $\eta \to 0$, we can $j)$, $1 \le j \le k$, such that setting $g_j = h_{\ell(j)}$ in [4.5] - [4.8] makes quantities on the righthand side there less than η. Thus

$$\lim_{\substack{\to 0 \\ \varepsilon A}} E[|Z_{\delta ij} - Z_{\alpha ij}|^2] \le 2\eta + \lim_{\substack{\delta, \alpha \to 0 \\ \delta, \alpha \varepsilon A}} E[|X_{\delta ij\ell(i)} - X_{\alpha ij\ell(i)}|^2] = 2\eta.$$

$_j$ converge in mean square for each i, j as $\delta \to 0$ in A. It follows : joint laws of $\{\phi_{\delta i}(t_j)(f_j), 1 \le i \le 4, 1 \le j \le k\}$ converge. Theorem |, now implies that $[\phi_{\delta 1}(\cdot), \ldots\ldots, \phi_{\delta 4}(\cdot)]$ converge as |; Q′))^4-valued random variables. Let $[\phi_1(\cdot), \phi_2(\cdot), \phi_3(\cdot), \phi_4(\cdot)]$:ts limit in law (abbreviated as "l.i.l" henceforth). By taking .l. in [3.2] along an appropriate subsequence,

$$\phi_1(t) = \phi_1(0) + \sum_{i=2}^{4} \phi_i(t) \quad \text{a.s.} \qquad [4.9]$$

<u>Theorem 4.1</u> $\phi_1(\cdot)$ is the $(\phi^4)_2$ process.

<u>Proof</u> We prove the theorem by identifying each term of [4.9]. Let $f\varepsilon Q$. By Jensen's inequality and stationarity, $E[|\int_0^t \phi_\delta(s)(C_\delta^{-\varepsilon} f)\,ds|^2] - \int_0^t \phi_\delta(s)(C^{-\varepsilon} f)\,ds|^2] \le t\,E[|\phi_\delta(0)(C_\delta^{-\varepsilon} f - C^{-\varepsilon} f)|^2] \le t\,K\|C_\delta^{-\varepsilon} f - C^{-\varepsilon} f\|_2^2$.

The righthand side tends to zero as $\delta \to 0$ by arguments similar to those employed in the proof of Lemma 4.1. Thus

$$\underset{\delta \to 0}{\text{l.i.l.}}\ (\phi_{\delta 1}(\cdot), \phi_{\delta 2}(t)(f)) = (\phi_1(\cdot), -2\,\phi_2(t)(f))$$
$$= \underset{\delta \to 0}{\text{l.i.l}}\ (\phi_\delta(\cdot), \int_0^t \phi_\delta(s)(C_\delta^{-\varepsilon} f)\,ds)$$
$$= \underset{\delta \to 0}{\text{l.i.l.}}(\phi_\delta(\cdot), \int_0^t \phi_\delta(s)(C^{-\varepsilon} f)\,ds)$$
$$= (\phi_1(\cdot), \int_0^t \phi_1(s)(C^{-\varepsilon} f)\,ds).$$

It follows that

$$\phi_2(t)(f) = \frac{1}{2} \int_0^t \phi_1(s)(C^{-\varepsilon} f)\,ds \quad \text{a.s.}$$

Similarly

$$E[|\int_0^t :\phi_\delta^3(s):_\delta (C_\delta^{1-\varepsilon} f)\,ds - \int_0^t :\phi_\delta^3(s):_\delta (C^{1-\varepsilon} f)\,ds|^2]$$

$\le t\,E[|:\phi_\delta^3(0):(C_\delta^{1-\varepsilon} f - C^{1-\varepsilon})|^2] \to 0$ as in $\delta \to 0$ in A, by arguments analogous to those above. Hence

$$\underset{\delta \to 0}{\text{l.i.l.}}\ (\phi_\delta(\cdot), \int_0^t :\phi_\delta^3(s):_\delta (C_\delta^{1-\varepsilon} f)\,ds) = (\phi_1(\cdot), -2\phi_3(t)(f))$$
$$= \underset{\delta \to 0}{\text{l.i.l.}}\ (\phi_\delta(\cdot), \int_0^t :\phi_\delta^3(s):_\delta (C^{1-\varepsilon} f)\,ds) \qquad [4.10]$$

Let $\alpha > \delta$ in A. Then

$$E[|\int_0^t :\phi_\delta^3(s):_\delta (C^{1-\varepsilon} f)\,ds - \int_0^t :\phi_\alpha^3(s):_\delta (C^{1-\varepsilon} f)\,ds|^2]$$

$\le t\,E[|:\phi_\delta^3(0):_\delta (C^{1-\varepsilon} f) - :\phi_\alpha^3(0):_\delta (C^{1-\varepsilon} f)|^2] \le 0(\alpha^\beta)$ for a suitable $\beta > 0$ uniformly in δ as $\delta \to 0$, by virtue of (9.6.9), p. 228, [4]. Thus the righthand side of [4.10] equals

$$\underset{\alpha \to 0}{\text{l.i.l.}}\ \underset{\delta \to 0}{\text{l.i.l.}}\ (\phi_\delta(\cdot), \int_0^t :\phi_\alpha^3(s):_\delta (C^{1-\varepsilon} f)\,ds)$$
$$= \underset{\alpha \to 0}{\text{l.i.l.}}\ (\phi_1(\cdot), \int_0^t :\bar{\phi}_\alpha^3(s):(C^{1-\varepsilon} f)\,ds)$$

where $\bar{\phi}_\alpha(\cdot)$ is defined by

$$\bar{\phi}_\alpha(t)(h) = \sum^\alpha \phi_1(t)(e_k) \langle e_k, h \rangle, \quad h\varepsilon Q.$$

The above limit equals

$$(\phi_1(\cdot), \int_0^t :\phi_1^3(s): (C^{1-\varepsilon} f)\, ds),$$

Thus

$$\phi_3(t)(f) = -\frac{1}{2}\int_0^t :\phi_1^3(s): (C^{1-\varepsilon} f)\, ds \text{ a.s.}$$

Finally, it is easy to check that $\phi_4(\cdot)$ will be a Wiener process with covariance $C^{1-\varepsilon}$. Thus $\phi_1(\cdot)$ satisfies [3.2] with initial law μ. By the uniqueness in law of this equation (proved in [2], Section IV), we conclude that $\phi_1(\cdot)$ is the $(\phi^4)_2$ process. QED

Corrollary 4.2 $\phi_\delta(\cdot)$ converge in law to $\phi(\cdot)$ as $C([0,\infty]; Q'$-valued random variables as $\delta \to 0$ in A, as defined originally.

Proof A careful look at the foregoing shows that any subsequence of A will have a further subsequence along which the above convergence holds. QED

ACKNOWLEDGEMENTS

This work was done while both of us were at the Scuola Normale Superiore, Pisa. Vivek S. Borkar would like to thank C.I.R.M., Italy, for travel support, and the Scuola Normale Superiore for financial support, which made this visit possible.

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THE SUPPORT OF THE DENSITY OF A FILTER IN THE UNCORRELATED CASE.

Mireille CHALEYAT-MAUREL
Université Paris VI,
Laboratoire de Probabilités,
Tour 56, 4 place Jussieu,
75230 Paris Cedex 05, France.

Dominique MICHEL
U.A.-C.N.R.S. 745,
Laboratoire de Statistique,
118, route de Narbonne,
31062 Toulouse Cedex, France.

The purpose of support theorems for stochastic processes is the description of the support of the law of a process (on its natural state space) by means of a set of solutions of a controlled system.

It was initiated by the celebrated theorem of D. Stroock and S. Varadhan [8]. They proved that the support of the law of a diffusion process, solution of a stochastic differential equation (written in Stratonovitch form) :

$$\begin{cases} dx_t = X_0(x_t)\, dt + X_i(x_t) \circ dw_t^i \\ x_0 \text{ fixed} \end{cases}$$

can be described as the closure (for the natural Banach topology on $C([0,1],\mathbb{R}^n)$) of the set of solutions of the following controlled system :

$$\begin{cases} \dfrac{dx_t^u}{dt} = X_0(x_t^u) + X_i(x_t^u)\, \dot{u}_t^i & u \in H^1([0,1],\mathbb{R}^d) \\ x_0^u = x \end{cases}$$

In [3] and [4], we followed this route and we proved a similar result for the unnormalized filter in the theory of correlated filtering of diffusions. More precisely, if φ is a C^∞- function from $\mathbb{R}^n$ to $\mathbb{R}$ with suitable growth conditions, the unnormalized filter associated to φ, $\rho_t\varphi$, is viewed as a stochastic process $t \to \rho_t\varphi$ taking values in $C^\infty(\mathbb{R}^n,\mathbb{R})$. We described the support of the law of $\rho_.\varphi$ on $C([0,1],C^\infty(\mathbb{R}^n,\mathbb{R}))$ endowed with its natural Frechet topology as the closure of a controlled partial differential equation (in the weak sense).

Here, we consider an uncorrelated situation and we suppose that