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THEORY OF INEQUALITIES AND THE CONTROLLABILITY OF LINEAR SYSTEMS

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The primary objectives of this paper are twofold: 1) to present a theory of controllability applicable to both finite and infinite dimensional linear control systems and 2) to show the relationship between the theory of inequalities of fundamental importance in mathematical programming, and controllability which is of fundamental importance in systems theory. It extends and generalises earlier results of Antosiewicz (1) and Conti (2).

Let X be a linear space with a locally convex Hausdorff topology $\mathcal{T}$ and let X^* be its topological dual. The natural pairing of X and X^* is represented as $\langle X,X^*\rangle$ and the fixed bi-linear functional by $\langle x,x^*\rangle$. The weak topology w for the linear space X with locally convex Hausdorff topology is the topology $w(X,X^*)$ of the natural pairing of X and X^* and the weak* topology for X^* is the topology $w(X^*,X)$ of the natural pairing of X^* and X [(3), Chapter 5: Sections 16 & 17].

Let U and X be linear speces with locally convex Hausdorff Topologies. Let $\langle U,U^*\rangle$ and $\langle X,X^*\rangle$ be their natural pairings. Let $T : U \to X$ be a continuous linear transformation and let $T^* : X^* \to U^*$ be the adjoint linear transformation.

<u>Definition 1</u> [(4), p. 246]: The polar A° in X^* of a subset A of X is defined by $A^\circ = \{x^* \in X^* : \langle x,x^*\rangle \leq 1\}$.

If A is equilibré then A° is also given by

$$A^{o} = \{x^{*} \in X^{*} : |\langle x, x^{*} \rangle| \leq 1\} .$$

Let U and X be locally convex Hausdorff topological spaces. U is to be thought of as the control space and X the state space of a control system. Consider a linear control system described by the operator equation

$$x = x_o + T(u) , \tag{1}$$

where $x \in X$, $u \in U$, $x_o \in X$ is a fixed element and $T : U \to X$ is a continuous linear transformation. A wide variety of linear systems may be described by this abstract model.

As an example, we shall consider a linear differential system in a Banach space. Let U and X be real Banach Spaces.

Let $t > 0$ and $1 < p \leq \infty$. We define $L_t^p(U)$ to be the Banach Space of all U-valued strongly measurable functions defined on $[0, t]$ such that

$$\int_0^t \| u(\tau) \|_U^p \, d\tau < \infty \quad \text{if} \quad 1 < p < \infty$$

and ess. sup. $\{ \|u(\tau)\|_U , 0 \leq \tau \leq t\} < \infty$ if $p = \infty$.

The Banach Space $L_t^p(U)$ is normed by,

$$\|u\|_p = \left(\int_0^t \|u(\tau)\|_U^p \, d\tau \right)^{\frac{1}{p}} , \quad 1 < p < \infty$$

and $\|u\|_\infty = \text{ess. sup} \{ \|u(\tau)\|_U , 0 \leq \tau \leq t\}$ if $p = \infty$.

In the sequel we shall assume that the space $L_t^p(U)$, $1 < p < \infty$ to be reflexive. This will be the case if U is reflexive and separable or uniformly convex. The dual space $[L_t^p(U)]^*$ is isometrically isomorphic to $L_t^q(U^*)$, $\frac{1}{p} + \frac{1}{q} = 1$.

Consider the linear differential system,

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2}$$

with initial condition $x(0) = x_o \in X$, where A is a linear closed operator with domain $D(A)$ which is the infinitesimal generator of a strongly continuous semi-group $T(t)$, $t \geq 0$, of linear bounded operators and B is a linear bounded operator mapping U into $D(A)$. We shall say that $x(\cdot)$ is a solution of Eq. (2) with initial condition $x(0) = x_o \in X$ if $x(\cdot)$ satisfies the integral equation

$$x(t) = T_t(x_o) + \int_0^t T_{t-\tau} \, Bu(\tau) \, d\tau \tag{3}$$

where the integral in the right hand side of Eq. (3) is in the sense of Bochner [(5), Section 3.7, p. 78].

For each $t > 0$, define a linear bounded transformation

$$R_t : u \to \int_0^t T_{t-\tau} \, Bu(\tau) \, d\tau$$

from $L_t^p(U)$ into X. Then Eq. (3) can be written as

$$x(t) = T_t(x_o) + R_t(u) \tag{4}$$

which fits our abstract model Eq. (1).

Necessary and Sufficient Conditions for Controllability

Consider the linear control system described by Eq. (1). Let $K \subseteq X$ be a closed convex set containing the null element and let $\Omega \subset U$ be a convex set which is compact with respect to the $w(U,U^*)$ - topology of U. Let K^o be the polar of K.

Definition 2: The system described by Eq. (1) is said to be controllable with respect to (x_o, Ω, K) if there exists a $\bar{u} \in \Omega$ such that $\bar{x} = x_o + T\bar{u} \in K$.

Our basic result is,

Theorem 1: The system described by Eq. (1) is controllable with respect to (x_o,Ω,K) if and only if

$$\langle x_o,x^*\rangle - 1 \leq \mathrm{Max}[\langle u,T^*x^*\rangle : u\in\Omega] \ , \quad \forall\, x^*\in K^o \qquad (5)$$

The proof of the theorem will proceed via Propositions 1-4.

<u>Proposition 1</u>: The system described by Eq. (1) is controllable with respect to (x_o,Ω,K) if and only if there exists a $\bar{u}\in\Omega$ such that

$$\langle x_o + T\bar{u}, x^*\rangle - 1 \leq 0 \ , \quad \forall\, x^*\in K^o \ . \qquad (6)$$

<u>Proof</u>: Since $K\subseteq X$ is a closed convex set in a locally convex Hausdorff topological space it is closed with respect to the $w(X,X^*)$ - topology of X [(3), Chapter 5, Proposition 17.1]. Since K contains the null element, by the Bipolar theorem [(4), p. 248], $K = (K^o)^o$. Then from the definition of a polar set, $x_o + T\bar{u}\in K$ if and only if

$$\langle x_o + T\bar{u}, x^*\rangle \leq 1 \ , \quad \forall\, x^*\in K^o \ .$$

The problem of controllability has thus been reduced to finding necessary and sufficient conditions for a feasible solution for an infinite system of linear inequalities to exist.

Let (X,T_1) be a real Hausdorff topological vector space and let R be the space of reals.

<u>Definition 3</u>: Let $C\subseteq X$ be a convex set. A function $f : C\to R$ is said to be <u>quasi-convex</u> on C if for any $r\in R$, the set

$$K = \{x\in C : f(x) \leq r\} \text{ is convex.}$$

The function f is said to be <u>quasi-concave</u> if the inequality is reversed.

The functions are said to be <u>strictly quasi-convex</u> (quasi-concave) if the inequalities are strict.

In a recent note Ky Fan (6) has proved certain geometric theorems regarding quasi-concave lower-semicontinuous functions. For our purposes we state one of the theorems in the following specialised form

<u>Proposition 2</u>: Let C_1,C_2 be non-empty compact convex subsets in a real Hausdorff topological vector space

X. Let $f_1 : C_1 \times C_2 \to R$ and $f_2 : C_1\times C_2\to R$ satisfy the following conditions:

(1) For every fixed $x_1\in C_1$, $f_1(x_1,x_2)$ is a lower-semicontinuous function on C_2 and for every fixed $x_2\in C_2$, $f_2(x_1,x_2)$ is a lower-semicontinuous function on C_1

(2) For every fixed $x_2\in C_2$, $f_1(x_1,x_2)$ is a quasi-concave function on C_1 and for every fixed $x_1\in C_1$, $f_2(x_1,x_2)$ is a quasi-concave function on C_2.

Let $r_1,r_2\in R$. If for every $x_2\in C_2$, there exists an $x_1\in C_1$ such that $f_1(x_1,x_2) > r_1$ and for every $x_1\in C_1$, there exists an $x_2\in C_2$ such that $f_2(x_1,x_2) > r_2$ then there exists a point $(\hat{x}_1,\hat{x}_2)\in C_1\times C_2$ such that

$$f_1(\hat{x}_1,\hat{x}_2) > r_1$$

and

$$f_2(\hat{x}_1,\hat{x}_2) > r_2 \ .$$

<u>Proposition 3</u>: Let C be a compact convex subset of a Hausdorff real topological vector space X. Let

$$P_n = \{\alpha = (\alpha_1, \ldots \alpha_n) : \alpha_i \geq 0 \ , i = 1,2,..n \ , \sum_{i=1}^{n} \alpha_i = 1\} \ .$$

Let g_i, $i = 1,2, \ldots n$ be a family of real-valued lower-semicontinuous functions on C such that

$$\sum_{i=1}^{n} \alpha_i\, g_i(x)$$

is quasi-convex on C for all $\alpha = (\alpha_1,\ldots\alpha_n)\in P_n$. Then the system of inequalities,

$$g_i(x) \leq 0 \ , \quad i = 1, 2, \ldots n \qquad (7)$$

has a solution in C if and only if there exists an $x\in C$ such that

$$\sum_{i=1}^{n} \alpha_i\, g_i(x) \leq 0 \ , \qquad (8)$$

for all $\alpha = (\alpha_1, \dots \alpha_n) \in P_n$.

Proof: The only if part is trivial.

To prove the if part, define two functions $\phi : C \times P_n \to R$ and $\psi : C \times P_n \to R$ by,

$$\phi(x,\alpha) = -\psi(x,\alpha) = -\sum_{i=1}^{n} \alpha_i \, g_i(x) \tag{9}$$

From the hypothesis of the theorem ϕ is quasi-concave on C . Also, clearly ϕ is a lower-semicontinuous function of α and ψ is a lower-semicontinuous function of x and a concave function of α . From condition Eq. (8) of the theorem, for any $\varepsilon > 0$ and any $\alpha \in P_n$ there exists an $x \in C$ such that $\phi(x,\alpha) > -\varepsilon$. Since $\phi(x,\alpha) = -\psi(x,\alpha)$ the two inequalities $\phi(x,\alpha) > -\varepsilon$ and $\psi(x,\alpha) > \varepsilon$ cannot be simultaneously satisfied. Hence by Proposition 2, for every $\varepsilon > 0$, there exists an $x_\varepsilon \in C$ such that $\psi(x,\alpha) \le \varepsilon$, $\forall\, \alpha \in P_n$.

For a given $\varepsilon > 0$, consider the set $C_\varepsilon = \{x \in C : g_i(x) \le \varepsilon$, $i = 1, 2, \dots n\}$. If we can prove that $\bigcap_{\varepsilon>0} C_\varepsilon \ne \emptyset$, where $\emptyset$ is the empty set, then the theorem is proved.

Since we have shown that for every $\varepsilon > 0$, there exists an $x_\varepsilon \in C$ such that $\psi(x_\varepsilon,\alpha) \le \varepsilon$, $\forall\, \alpha \in P_n$, putting

$$\alpha = (1, 0, \dots 0) , \quad (0, 1, 0, \dots 0)$$

etc. successively we obtain

$$g_i(x_\varepsilon) \le \varepsilon , \quad i = 1, 2, \dots n .$$

Hence $C_\varepsilon = \{x \in C : g_i(x) \le \varepsilon$ is non-empty. Since each g_i has been assumed to be lower-semicontinuous C_ε is closed. C_ε is also bounded and hence it is compact. We also have for $0 < \varepsilon' < \varepsilon$, $C_{\varepsilon'} \subset C_\varepsilon$ and hence $\bigcap_{\varepsilon>0} C_\varepsilon \ne \emptyset$ [(4), p. 21]. Hence there exists an $x \in C$ such that Eq. (7) is satisfied.

Remark: If there exists an $x \in C$ such that Eq. (7) is satisfied, then there exists an $\hat{x} \in C$ such that

$$\min_{x \in C} \sum_{i=1}^{n} \alpha_i g_i(x) = \sum_{i=1}^{n} \alpha_i g_i(\hat{x}) \le 0 , \quad \forall\, \alpha = (\alpha_1, \dots \alpha_n) \in P_n$$

Proposition 4: Let C be a compact convex subset of a real Hausdorff topological vector space X . Let I be an index set. For any $k > 0$, $I_k \subset I$ is the set $I_k = \{i : i = 1, 2, \dots k\}$. Let $\{g_i\}_{i \in I}$ be an infinite family of real-valued lower-semicontinuous functions on C such that for any $k > 0$

$$\sum_{i \in I_k} \alpha_i \, g_i(x)$$

is quasi-convex for all $\alpha \in P_k$. Then the system of inequalities,

$$g_i(x) \le 0 , \quad i \in I \tag{10}$$

has a solution in C , if and only if for any finite set of indices $k_1, k_2 \dots k_n \in I$ and for any $\alpha \in P_n$, there exists an $x \in C$ such that

$$\sum_{i=1}^{n} \alpha_i \, g_{k_i}(x) \le 0 . \tag{11}$$

The proof of the theorem is omitted for the sake of brevity. Analogues of these theorems for finite dimensional spaces may be found in Berge (7).

Proof of Theorem 1

By assumption $\Omega \subset U$ is a convex set which is $w(U,U^*)$ - compact. The left hand side of Eq. (6) is continuous on Ω and hence it is $w(U,U^*)$ - continuous on Ω [(3), p. 154, Proposition 17.3]. Hence from Propositions 3 and 4 there exists a $u \in \Omega$ such that $x_o + T\,u \in K$ if and only if

$$\min_{u \in \Omega} \sum_{i=1}^{n} \alpha_i [\langle x_o, x_i^* \rangle + \langle u, T^*(x_i^*) \rangle - 1] \le 0 \tag{12}$$

holds for any finite number of $x_i^* \in K^o$ and for any $\alpha = (\alpha_1, \dots \alpha_n) \in P_n$. But

$$\sum_{i=1}^{n} \alpha_i [\langle x_o, x_i^* \rangle + \langle u, T^*(x_i^*) \rangle - 1]$$

$$= \sum_{i=1}^{n} [\langle x_o, \alpha_i x_i^* \rangle + \langle u, T^*(\alpha_i x_i^*) \rangle - 1$$

$$= [\langle x_o, \sum_{i=1}^{n} \alpha_i x_i^* \rangle + \langle u, T^*(\sum_{i=1}^{n} \alpha_i x_i) \rangle - 1]$$

Since K^o is convex, putting

$$x^* = \sum_{i=1}^{n} \alpha_i x_i^* \in K^o ,$$

we obtain the desired result.

As a direct consequence of Theorem 1 the existance of minimum norm and best approximate controls can be established and moreover expressions for the minimum norm and minimum miss distance can be obtained in terms of the system data.

Controllability in the Presence of a Disturbance

Let U and W be reflexive Banach spaces and let X be a Hilbert Space. Consider the linear system

$$x = x_o + T(u) + S(w) \tag{13}$$

where $T : U \to X$ and $S : W \to X$ are continuous linear transformations and x_o is a given element in X. w is to be thought as a disturbance acting on the system. Let the controller restraint set be $\Omega_U = \{u \in U : ||u||_U \le \rho\}$ and the disturbance set be

$$\Omega_d = \{d \in X : d = Sw, ||Sw||_X \le \sigma\} .$$

Let the target set be

$$K = \{x \in X : ||x - x_d||_X \le \varepsilon\}$$

where x_d is the null element in X.

<u>Theorem 2</u>: Let the system (Eq. (13)) be controllable with respect to (x_o, Ω, K) in the absence of disturbances. Then the perturbed system is controllable with respect to (x_o, Ω, K) if and only if

$$\sigma \le \varepsilon - \text{Max}[0, \text{Sup}\{ |(x_d - x_o | x^*)| - \rho \, ||T^*(x^*)||_{U^*} : ||x^*||_{X^*} = 1\}] .$$

<u>Proof</u>: The perturbed system is controllable if and only if

$$\underset{u \in \Omega_u}{\text{Min}} \; \underset{d \in \Omega_d}{\text{Max}} [||Tu + d - (x_d - x_o)||_X] \le \varepsilon \tag{14}$$

$$|| Tu + d - (x_d - x_o) ||_X \le ||d||_X + ||Tu - (x_d - x_o)||_X$$
$$\le \sigma + ||Tu - (x_d - x_o)||_X$$

Let

$$\hat{d} = \frac{\sigma(Tu + x_o - x_d)}{|| Tu + x_o - x_d ||_X} .$$

It is clear that $\hat{d}$ is the maximising d in Eq. (14) and

$$\underset{u \in \Omega_u}{\text{Max}} [||Tu + d - (x_d - x_o)||_X] = \sigma + ||Tu + x_o - x_d||_X .$$

Hence the perturbed system is controllable if and only if

$$\underset{u \;\; \Omega_u}{\text{Min}} [||Tu + x_o - x_d||_X] + \sigma \le \varepsilon .$$

From the controllability conditions it may be shown that

$$\underset{u \in \Omega_u}{\text{Min}} [||Tu + x_o - x_d||_X] = \text{Max}[0, \text{Sup}\{ |(x_d - x_o | x^*)| - \rho \, ||T^*(x^*)||_{U^*}\}]$$

and hence the theorem is proved.

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ON THE EXISTENCE OF SATISFACTORY CONTROL

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Study of the sensitivity of optimal control have indicated a need for ensuring a sufficiently small variation of the performance over a given range of variations of the disturbances. This is provided by the so-called satisfaction approach to the control problem in which the objective consists in finding a control which will ensure a satisfactory performance over a range of disturbances.

In the present paper we shall give some conditions for the existance of the satisfactory control and will also indicate how these conditions can be used in a constructive way for a procedure to arrive at a strategy leading to a satisfactory control.

Consider at first the problem of controlling a system with disturbances.

Let X_1, X_2 and X_3 be real normed linear spaces and M, U and Y are subsets of X_1, X_2 and X_3, respectively. Let the system S be a mapping $S: M \times U \rightarrow Y$ and the behavior of S be evaluated by the performance functional $G: M \times Y \rightarrow R$ where R is the real line. Furthermore, let

$$S_m = S|\{m\} \times U \quad ; \quad G_m = G|\{m\} \times U$$

where $m \in M$. A mapping H_m can be then defined

$$H_m: U \rightarrow R, \text{ such that}$$

$$H_m(u) = G_m(S_m(u))$$