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# Control of LQG Systems Under Communication Constraints

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**Abstract**

We consider the control performance of an LQG system with a noisy analog feedback channel between the state-observation and the controller. To bound the performance, we use the sequential rate distortion function and the assumption of equi-memory. We then discuss the tradeoffs between control and communication costs and how to relax the equi-memory assumption.

**Keywords:** Sequential Rate Distortion, LQG Control, Communication Constraints, Relaxing Equi-memory, Control/Communication Tradeoff.

## 1 Introduction

Many modern control systems are employing multiple sensors and actuators that are geographically distributed. There are many issues of coordination and communication that must be dealt with. It becomes important then to determine what the sensors should transmit to the controller and what channel rates are required to achieve a specified performance.

In this paper we examine the classical LQG control problem under communication constraints. We can view this as a traditional control problem, except that we must design the observation equation subject to some constraints on the observation alphabet and power.[2] It will be shown that the unstable eigenvalues of the plant are intimately related to the capacity requirements for the communication channel — if the the rate is less than some threshold, then the cost is necessarily infinity as it is impossible to even stabilize the system.

Such thresholding behavior reminds one of information theory where Shannon introduced the distortion rate framework. He showed that one can achieve entropy rates arbitrarily close to the rate-distortion function for suitably long lossy block codes. Unfortunately, long block codes imply long delays in communication systems, which are unacceptable in control applications. To remedy this, we will introduce the sequential rate distortion function and use it to derive bounds on the performance of any LQG control system that involves a rate-limited communication channel.

This paper is organized as follows: in section 2 we describe the model. In section 3 we describe the communication channel. In section 4 we discuss the encoders and their properties. In section 5 we describe the sequential coding results we will need. In section 6 we put the pieces together to bound the LQG performance assuming that we have equi-memory. In section 7 we look more closely at the role of equi-memory and discuss how it can be relaxed by reinterpreting the cost on control actions.

## 2 The Control Problem

We consider the following control system:

$$X_{k+1} = AX_k + BU_k + W_k, \quad k \geq 0 \tag{1}$$

where $\{X_k\}$ is an $\mathbb{R}^d$-valued state process. $X_0$ is a zero mean Gaussian with variance $K_{X_0}$. $\{U_k\}$ is an $\mathbb{R}^m$-valued control process and $\{W_k\}$ is an $d$-dimensional IID Gaussian process with zero mean and variance $K_W$. $A \in \mathbb{R}^{d\times d}, B \in \mathbb{R}^{d\times m}$ and $(A, B)$ is controllable.

Our objective function is the average infinite horizon

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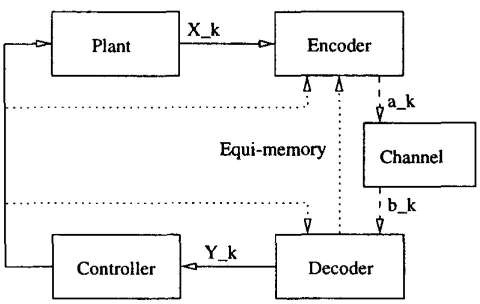
quadratic cost:

$$\limsup_{N\to\infty} \frac{1}{N} E\left( X_N' Q X_N + \sum_{i=0}^{N-1} (X_k' Q X_k + U_k' R U_k) \right) \tag{2}$$

Where $Q \in \mathbb{R}^{d\times d}$ is symmetric positive semidefinite, $R \in \mathbb{R}^{m\times m}$ is symmetric positive definite, and $(A, Q^{\frac{1}{2}})$ is observable. These are the standard LQG assumptions and can be found, for instance, in [4].

In addition, we have a communication channel with inputs $a_k \in \mathcal{A}$ and outputs $b_k \in \mathcal{B}$. Our channel model is covered in section 3.




**Definition 2.1** *The information state at time $k$ is $I_k = (X_0^{k-1}, a_0^{k-1}, b_0^{k-1}, Y_0^{k-1}, U_0^{k-1}) \in \mathcal{I}_k$ where $X_k, a_k, b_k, Y_k, U_k$ are the state, channel input, channel output, decoder output, and control signal, respectively. $\mathcal{I}_k$ is called the information space.*

We assume all the primitive random variables $\{X_0, W_k, V_k, k \geq 0\}$ are defined on a common probability space and are independent of each other. After specification of the control law, encoder law, and decoder law the information state, $I_k$, becomes a well-defined random variable.[13]

**Definition 2.2** *The information space for the encoder at time $k$ is $\mathcal{I}_k^{\mathcal{E}}$ which is a coordinate projection of $\mathcal{I}_k$. The information state of the encoder at time $k$ is $I_k^{\mathcal{E}} \in \mathcal{I}_k^{\mathcal{E}}$. The encoder, $\mathcal{E}$, at time $k$ is the map:*

$$\mathcal{E}_k : \mathcal{I}_k^{\mathcal{E}} \times \mathbb{R}^d \to \mathcal{A}$$

*where $\mathcal{E}_k$ takes $(I_k^{\mathcal{E}}, X_k) \mapsto a_k$.*

Similarly, the decoder and controller are $\mathcal{D}_k : (I_k^{\mathcal{D}}, b_k) \mapsto Y_k$ and $\mathcal{U}_k : (I_{\mathcal{U},k}, Y_k) \mapsto U_k$. Specification of all these information spaces is a part of the problem, and the actual encoder, decoder, and controller maps themselves are the solution.

In general this sort of distributed control problem is very difficult. There are, though, certain information spaces that allow for tractable solutions [13].

## 3 Communication Channels

Clearly the choice of communication channel effects the design of the optimal controller and encoder/decoder pair. One possible model is that of a a noiseless, delay-free, digital channel. We consider this model in [12].

Here we describe an analog additive white Gaussian noise (AWGN) channel with a power constraint.

**Definition 3.1** *An AWGN channel is an analog channel modeled as*

$$b_k = a_k + v_k$$

*where $\mathcal{A} = \mathcal{B} = \mathbb{R}^d$, and $\{v_k\}$ is an IID Gaussian process with zero mean and variance $K_V$ representing the channel noise.*

The information state and space of the encoder $\mathcal{E}_k$ at time $k$ are $I_k^{\mathcal{E}} \triangleq (X_0^{k-1}, U_0^{k-1}, a_0^{k-1})$ and $\mathcal{I}_k^{\mathcal{E}} = (\mathbb{R}^d)^k \times (\mathbb{R}^m)^k \times \mathcal{A}^k$ respectively. Similarly for the decoder $\mathcal{D}_k$, we have $I_k^{\mathcal{D}} \triangleq (Y_0^{k-1}, U_0^{k-1}, b_0^{k-1})$ and $\mathcal{I}_k^{\mathcal{D}} = (\mathbb{R}^d)^k \times (\mathbb{R}^m)^k \times \mathcal{B}^k$ respectively. Note that the information we have at time $k+1$ contains the information we had at time $k$.

Furthermore, to prevent degenerate solutions, we impose the following power constraint:

$$E(\|a_k\|^2) < d \times P$$

for some total power $P$ per time step. Shannon's classical theorem shows that the maximum achievable rate $R$ of this channel is

$$R = \max_{tr(K_{A,k}) \leq d\times P} \frac{1}{2d} \log \frac{|K_{A,k} + K_V|}{|K_V|}$$

where $K_{A,k} = cov(a_k|\mathcal{I}_k^{\mathcal{E}})$ is the covariance matrix of the transmitted signal $a$. For a rate $R$ there is a unique power constraint $P$ and vice-versa.

## 4 Encoders and Decoders

### 4.1 Equi-memory

**Definition 4.1** *An encoder/decoder pair are said to be equi-memory if:*

1. *There exist two maps $\Gamma_k^1 : \mathcal{I}_k^{\mathcal{E}} \to \mathcal{I}_k^{\mathcal{D}}$ and $\Gamma_k^2 : \mathcal{I}_k^{\mathcal{D}} \to \mathcal{I}_k^{\mathcal{D}}$ such that $\Gamma_k^1(\mathcal{I}_k^{\mathcal{E}}) = \Gamma_k^2(\mathcal{I}_k^{\mathcal{D}})$ for all $k$.*
2. *The encoder has the structure: $\mathcal{E}_k(I_k^{\mathcal{E}}, X_k) = \mathcal{E}_k(\Gamma_k^1(I_k^{\mathcal{E}}), X_k)$ and the decoder has the structure: $\mathcal{D}_k(I_k^{\mathcal{D}}, b_k) = \mathcal{D}_k(\Gamma_k^2(I_k^{\mathcal{D}}), b_k)$ for all $k$.*

In order to satisfy the equi-memory condition, we require that the encoder's information state effectively contain the past decoder output signals, $Y_k$.[1] For discussion on how to relax the equi-memory condition see [7] and our discussion in section 7 of this paper.

### 4.2 Predictive Encoders

Predictive encoders are commonly used in practice when encoding sequential data [7]. These are encoders that transmit across the channel a coding of the error: the true state of the system minus the *decoder's* best prediction of the state. We need equi-memory because the encoder needs to know the state estimate at the decoder in order to compute this error.

**Definition 4.2** *A predictive encoder is an encoder*

$$a_k = \mathcal{E}_k(I_k^{\mathcal{E}}, X_k) = \mathcal{E}_k\left(X_k - \mathcal{P}_k^{\mathcal{E}}(I_k^{\mathcal{E}})\right)$$

*where the predictor is a map* $\mathcal{P}_k^{\mathcal{E}} : \mathcal{I}_k^{\mathcal{E}} \to \mathbb{R}^d$.

The predictors are chosen based on the objective cost. For the quadratic cost we use the following expectation predictor:

$$\mathcal{P}_k^{\mathcal{E}}(I_k^{\mathcal{E}}) = E\left(X_k | \Gamma_k^1\left(I_k^{\mathcal{E}}\right)\right)$$

The key reason for the equi-memory assumption is that with it, for expectation predictive encoder/decoder pairs, the error $e_k = X_k - Y_k$ is independent of the control. [12]

## 5 Sequential Lossy Coding

In order to lower bound the error in state-estimation across a communication channel, we will define the information theoretic sequential distortion rate function (SDR) function for Gauss-Markov processes:

$$X_{k+1} = AX_k + W_k \tag{3}$$

where $X_k \in \mathbb{R}^n$ and $W_k$ is an IID Gaussian process with mean zero and variance $K_W$. $X_0$ is Gaussian with mean zero and variance $K_{X_0}$. Notice that this is just equation (1) with the control set to zero.

Fix the single-letter difference distortion measure $d(x, \hat{x}) = (x - \hat{x})' M (x - \hat{x})$ where $M$ is symmetric and positive definite.

Our goal is to encode the reconstructions $\hat{X}_k$ to minimize

$$\limsup_{N\to\infty} \frac{1}{N} E\left\{\sum_{k=0}^{N-1} d(X_k, \hat{X}_k)\right\}$$

over all encoder/decoder pairs.

[1]While in general, this would require explicit noiseless feedback from the decoder to the encoder, in many cases, it may be possible for the encoder to infer this information if it has physical access to the control signals.

**Definition 5.1** *The mutual information between two random variables* $X, Y$ *with density* $p(X, Y)$ *and marginals* $p(x), p(y)$ *is defined as*
$I(X, Y) \triangleq \int p(x, y) \log \frac{p(x,y)}{p(x)p(y)} dxdy$.

Since the encoders which achieve the classical $D(R)$ curves ([3] gives the stationary case, in [11] we discuss generalizations to unstable processes) are not causal, we proceed to define the sequential distortion rate function (also called the prognostic epsilon entropy function [8].) Here the minimizing conditional law must be *sequential.* A conditional law $p(\hat{x}_1^n | x_1^n)$ is *sequential* if $\hat{X}_i$ is independent of $X_j \; \forall j > i$ given $X_1^i$. In order to get a lower bound, the idea is to "relax" the problem and optimize over conditional laws and not the deterministic quantizers.

**Definition 5.2**
$D_{N,Seq}(R) = \inf_{p(\hat{X}_1^N | X_1^N)} \frac{1}{N} Ed(X_1^n, \hat{X}_1^N)$ *such that* $\frac{1}{N} I(X_1^N; \hat{X}_1^N) \le R$ *and* $p(\hat{X}_1^N | X_1^N)$ *is sequential.*

**Definition 5.3** $D_{Seq}(R) = \limsup_{N\to\infty} D_{N,Seq}(R)$

The asymptotic performance of these functions in the vector and high rate case can be found in the upcoming paper [12].

For the scalar case we have:

$$D_{Seq}(R) = \begin{cases} \frac{K_W M}{2^{2R} - A^2} & \text{if } R > \log A \\ \infty & \text{if } R \le \log A \end{cases}$$

Note here that $R > \log A$ for the scalar reconstruction to have finite distortion. More generally, to have finite distortion we must transmit at least $R \ge \sum_{i=1}^d \max\{0, \log \sigma_i(A)\}$ bits every time step. The linear system can thus be thought of as "producing" this many bits of information at each time step.

## 6 Using Sequential Distortion Rate to Bound Performance

Now, we have all the pieces in place. It remains to choose the metric used for the quantization, and the control law. Clearly, the control signals will be a function of the past controls and decoder outputs — $I_k^{\mathcal{U}} \triangleq \{Y_0^{k-1}, U_0^{k-1}\}$.

**Lemma 6.1** *If the encoder and decoder are equi-memory and predictive then the information state of the decoder is a sufficient statistic for the state of the system.*

**Theorem 6.1** *The control problem separates into a state estimator and a certainty equivalent controller.*

The proof is given in [12] and relies critically on the above lemma. The optimal controller is

$$\mathcal{U}_k(Y_k) = -(B'KB + R)^{-1}B'KAY_k.$$

where $K$ satisfies the following Riccati equation

$$K = A'(K - KB(B'KB + R)^{-1}B'K)A + Q.$$

Furthermore, letting $\bar{D}$ be the squared-error distortion accrued by the encoder/decoder pair, it turns out that the total average cost is $tr(KK_W) + tr((A'KA - K + Q)\bar{D})$

$$\text{Thus, Ave. Cost} \geq tr(KK_W) + D_{Seq}(R) \quad (4)$$

where the sequential quantizer is optimized for the weight matrix $(A'KA - K + Q)$.

In the scalar case, we thus have:

$$\text{Ave. Cost} \geq \begin{cases} KK_W + \frac{K_W(A^2K-K+Q)}{2^{2R}-A^2} & \text{if } R > \log A \\ \infty & \text{if } R \leq \log A \end{cases}$$

When there is no cost on control, $R = 0$, we get $K = Q$ and for the scalar problem we get:

$$\text{Ave. Cost} \geq \begin{cases} \frac{K_W Q}{1-(\frac{A}{2^R})^2} & \text{if } R > \log A \\ \infty & \text{if } R \leq \log A \end{cases}$$

It turns out that we can get the bounds above to hold with equality if we use the AWGN channel.[9] It is "matched" to the problem at hand.

## 7 Equi-memory Reexamined

In some fashion the encoder must be able to continuously monitor the error between the true state and the decoder's estimate of the state. In order to understand the situation better, let us concentrate on the scalar case, with the power-constrained AWGN channel from the encoder to the controller. However, let us suppose that there is no direct connection between the controller and the encoder. The question arises, can we still stabilize the system while meeting the rate constraint on the channel?

We notice that if there is no cost on the control, there exists a $L$ such that $BL = -A$. This is a "minimum variance" controller. The encoder then observes the state $X_k = AX_{k-1} - AY_{k-1} + W_k = AE_{k-1} + W_k$. This is exactly what the expectation predictive encoder uses to compute the transmitted signal. Hence, we do not need any other link from the controller to the encoder — *the plant itself can act as the link.* Equi-memory becomes redundant.

### 7.1 Reinterpreting the cost on control — the plant as a "channel"

We now reconsider the role of the cost on the control. In a physical system, the values for the control variables and the plant state are in general fundamentally incomparable quantities expressed in different underlying units. So rather than viewing $R$ as an *a-priori* given cost term, it is more realistic to view it as filling the role of a Lagrange multiplier on a more fundamental underlying constraint of the form $E(U^2) \leq P_2$. For the purpose of simplicity, we will consider $U$ in $B$-units, thereby setting $B = 1$.

Noticing that the encoder has access to past values of $X$, we can see that it effectively observes $U_k + W_k$ where $U_k$ is subject to a power constraint and $W_k$ is white Gaussian noise. Thus, we can view the plant as an AWGN channel, and nominally define a rate $R_2 = \frac{1}{2}\log\left(1 + \frac{P_2}{K_W}\right)$. For this nominal rate $R_2$ to be meaningful, we must see whether it behaves like a "rate." Is there a minimum rate $R_2$ that we must have in order to have finite cost? Recall that we have already shown that $R_1 > \log A$ is required.

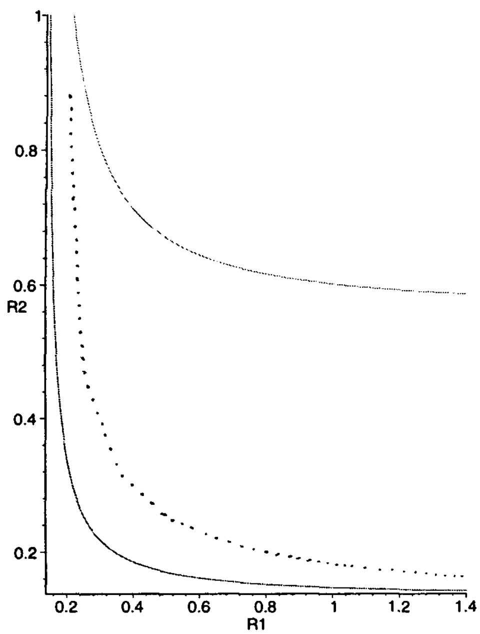
To find the minimum rate on $R_2$, we relax the constraint on $R_1$. By boosting the power/rate on the channel from the encoder to the controller, we approach $\hat{X}_k = X_k$ with no error. Since the optimal control law is linear, we can write $U_k = -l_2 X_k$. Thus, $E(U^2) = l_2^2 E(X^2)$. So, the limiting closed loop system is $X_{k+1} = (A - l_2)X_k + W_k$, and it is clear that $E(X^2) = \frac{K_W}{1-(A-l_2)^2}$. Thus, $E(U^2) = \frac{l_2^2 K_W}{1-(A-l_2)^2}$. We can minimize this expression by setting $l_2 = \frac{A^2-1}{A}$. So, $E(U^2)$ is bounded below by $(A^2 - 1)K_W$. This gives us $R_2 > \log A$ as a requirement for stability.

It is important to note that the situation is not completely symmetric between $R_1$ and $R_2$. While the performance strictly improves as we increase $R_1$, there is an upper bound for $R_2$. If we relax the constraint and optimize, we find that $l_2 = A$ is the choice that minimizes $E(X^2)$. We can compute $E(U^2)$ for the closed loop system (setting $K_W = 1 = K_V$ as a part of the choice of units), resulting in $E(U^2) = \frac{A^2-A^2 2^{-2R_1}}{1-A^2 2^{-2R_1}}$. Thus, $R_2 \leq \frac{1}{2}\log\left(\frac{A^2+1-2A^2 2^{-2R_1}}{1-A^2 2^{-2R_1}}\right)$.

This upper-bound value for $R_2$ also represents something else. If $R_2$ has this value, then the equi-memory assumption is no longer needed to achieve optimal performance. Thus, the role of the equi-memory assumption becomes more clear — it exists to allow us to encode the data on channel 1 without having to consider any rate limitation on the "channel" through the plant. In fact, given a fixed $R_1$ constraint, for scalar plants the signal sent through this channel depends only on the realization of the $W_k$ and $V_k$ processes. It does not depend on $R_2$ at all.

### 7.2 Trading off the two rates

Equimemory vs No Equimemory for A=1.1




In the above plot, we have plotted curves showing the $R_1$, $R_2$ tradeoff by marking the edge of stability (the lower bound on $R_2$) and also the "Minimum Variance" scheme (the upper bound on $R_2$). It is within the gap between these two that the role of equi-memory becomes significant.

To understand that role, we propose the explicit consideration of a third channel directly connecting the controller to the encoder. with rate constraint $R_3$. This is considerably more complicated, and we have only begun to understand its properties.[12] However, this is what we know so far:

- $R_3 = \infty$ is the equi-memory case.
- Even if $R_3 = 0$, the fundamental limits that $R_1 > \log A$ and $R_2 > \log A$ are tight since by letting one rate tend to infinity, we can let the other approach $\log A$.

A workable suboptimal scheme for $R_3 = 0$ can be obtained by fixing the controller and decoder as is, and using a Kalman filter at the encoder to estimate the decoder state from observations of $X$ and knowledge of past inputs to the channel between the encoder and decoder. In the plot, the faint line shows that such a scheme can stabilize the system without using too much additional rate relative to the equi-memory case. In general, the smaller $A$ is, the greater (proportionately) will be the gap between the no-equimemory scheme and the equimemory bound.

- The case of arbitrary $R_3$ requires some sort of nonlinear signaling scheme over channel 3.

Channels 2 and 3 together can be viewed as a vector channel with separate per-channel power constraints. We know that except in degenerate cases, no linear scheme can be optimal for such a situation.[10] In general, as $R_3$ gets larger, the performance curve will approach the equi-memory bound.[12]

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