MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF PHYSICS

Academic Programs
Room 4-315

Phone: (617) 253-4851
Fax: (617) 258-8319

DOCTORAL GENERAL EXAMINATION

PART II

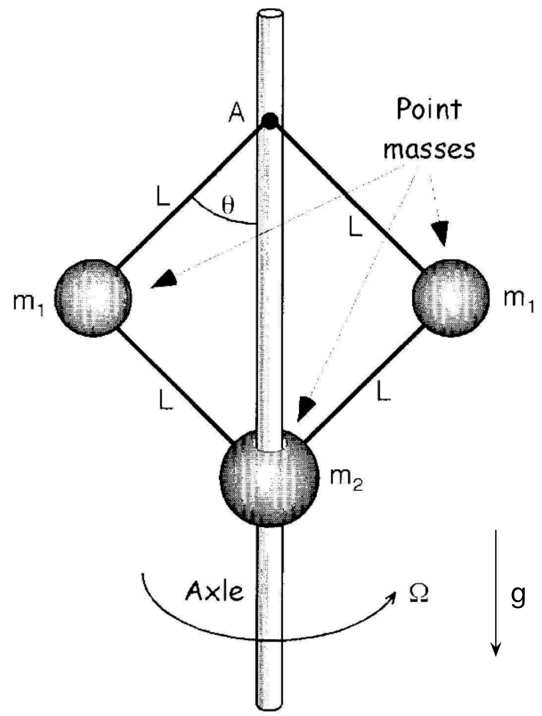
August 31, 2012

FIVE HOURS

1. This examination is divided into four sections, Mechanics, Electricity & Magnetism, Statistical Mechanics, and Quantum Mechanics, with two problems in each. Read both problems in each section carefully before making your choice. Submit ONLY one problem per section. IF YOU SUBMIT MORE THAN ONE PROBLEM FROM A SECTION, BOTH WILL BE GRADED, AND THE PROBLEM WITH THE LOWER SCORE WILL BE COUNTED.
2. Use a separate booklet for each problem. Write your name and the problem number (I.2 for example) on the front of each booklet.
3. Calculators may not be used.
4. No books or reference materials may be used.

SECTION I: CLASSICAL MECHANICS

Classical Mechanics 1: Equilibrium States of a Governor



Consider the motion of an idealized "governor", shown in the above figure. This mechanical device consists of three **point** masses, two with mass m_1 and a third with mass m_2. The masses are connected by **massless** rigid rods of length L which are free to pivot about all joints. At a point A shown in the figure, the upper two rods are attached to a vertical axle. There is a downward gravitational acceleration g. As the angle θ between the rods and the axle varies, the mass m_2 slides freely along the axle.

For this problem, the axle is brought to a certain angular speed $\dot{\phi} = \Omega$ which is then kept constant in time. [*Aside*: In a real mechanical governor, the height of the mass m_2 regulates the fuel intake to an engine in order to keep Ω constant.]

(a) (*4 pts*) The system settles into an equilibrium configuration with angle $\theta_{\rm eq}$. Find the corresponding angular speed Ω.

(b) (*2 pts*) What is the minimum rotation speed $\Omega_{\rm min}$ for which the equilibrium angle $\theta_{\rm eq}$ is non-zero? What is the limiting behavior of $\theta_{\rm eq}$ as $\Omega \to \infty$?

(c) (*4 pts*) For $\Omega > \Omega_{\rm min}$, what is the frequency ω of small oscillations about $\theta_{\rm eq}$? Express your answer in terms of m_1, m_2, Ω, and $\theta_{\rm eq}$.

Classical Mechanics 2: Relativistic spaceship

A spaceship ejects fuel with a constant exhaust velocity v_{ex} in its own frame, and travels in a straight line through empty space with negligible gravity. Initially the spaceship has a full fuel tank and a total rest mass M_0 (spaceship + fuel).

(a) (*2 pts*) Let S' be an inertial frame of reference in which the spaceship is instantaneously at rest, and let $M(t')$ denote the mass of the spaceship as a function of the time t' in this frame. Derive an expression for $a'(t')$, the acceleration of the spaceship in the frame S', at the time t' when it is at rest. The expression may involve $M(t')$ and any of its derivatives. Assume that the exhaust speed is non-relativistic: $v_{\text{ex}} \ll c$.

(b) (*3 pts*) The spaceship starts from rest in an inertial frame S and accelerates in the x-direction. The engines fire until all the fuel is consumed. With an empty fuel tank, the rest mass of the spaceship is M_0/f where $f > 1$ is a constant.

Assume that the spaceship achieves a **non-relativistic** final speed in frame S. Find this final speed and write your answer in terms of v_{ex} and f.

(c) (*5 pts*) Suppose the same conditions hold as in part (b), except that the spaceship achieves a **relativistic** speed in frame S. The exhaust velocity v_{ex} is still assumed to be non-relativistic. Find the final speed of the rocket in S in terms of v_{ex} and f, using special-relativistic kinematics and dynamics. Hint: you may want to relate the accelerations a' and a that are observed in frames S' and S respectively.

Possibly useful integrals:

$$\int \frac{ds}{(1-s^2)^{1/2}} = \sin^{-1}(s), \tag{1}$$

$$\int \frac{ds}{(1-s^2)} = \frac{1}{2}\ln\frac{1+s}{1-s}, \tag{2}$$

$$\int \frac{ds}{(1-s^2)^{3/2}} = \frac{s}{(1-s^2)^{1/2}}. \tag{3}$$

SECTION II: ELECTRICITY & MAGNETISM

Electromagnetism 1: Magnetic monopoles

Imagine there exist magnetic monopoles. The magnetic charge density ρ_m and magnetic current density $\vec{J}_m$ are analogous to the electric charge density ρ and electric current density $\vec{J}$. In particular, magnetic charge is conserved locally, just as electric charge is conserved. Maxwell's equations (in Gaussian units) are

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho, \quad \vec{\nabla} \cdot \vec{B} = 4\pi\rho_m, \quad \vec{\nabla} \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c}\vec{J}, \quad \vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} - \frac{4\pi}{c}\vec{J}_m. \quad (1)$$

(a) (*1 pt*) Consider a pointlike magnetic monopole with magnetic charge g. In quantum mechanics, the requirement that the wavefunction phase is single-valued leads to a constraint on the magnetic flux through any closed surface,

$$\frac{e}{\hbar c} \oint \vec{B} \cdot d\vec{S} = 2\pi n, \quad (2)$$

where n is an integer and $e > 0$ is the magnitude of the electron charge. From this condition, it follows that magnetic charge must be quantized, i.e., all magnetic charges are integral multiples of some elementary unit. Derive the elementary unit of magnetic charge in terms of fundamental constants.

(b) (*2 pts*) Consider a system of two point charges: a positron with charge e and a magnetic point charge g separated by a distance $2d$. Use the cylindrical coordinate system illustrated in Figure 1 (on the next page), in which the origin is at the midpoint between the charges, the positron is at $z = -d$ and the magnetic charge is at $z = +d$. Compute the electric and magnetic fields as a function of (r, θ, z).

(c) (*6 pts*) Compute the total angular momentum contained in the fields from part (c).

The following integrals may be useful:

$$\int \frac{x\ dx}{(x^2 + \beta x + \gamma)^{3/2}} = \left(\frac{1}{\beta^2/4 - \gamma}\right) \frac{\beta x/2 + \gamma}{(x^2 + \beta x + \gamma)^{1/2}} \quad (3)$$

$$\int_{-\infty}^{\infty} dx\ \frac{(a^2 + x^2) - \sqrt{(a^2 - x^2)}}{(a^2 + x^2)^2 - (a^2 - x^2)^2} = \frac{2}{a} \quad (4)$$

The following vector identity may be useful:

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}). \quad (5)$$

(d) (*1 pt*) Use the results from part (b) and (d), show that the angular momentum is also quantized and derive the elementary unit of angular momentum.

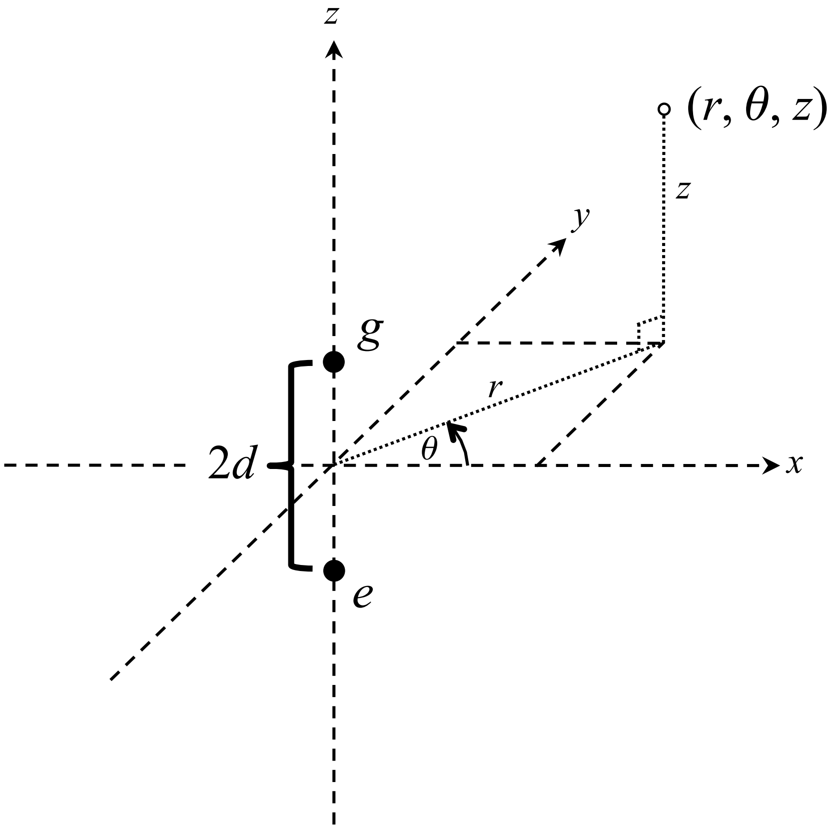


Figure 1: Coordinate system for problem E&M 1b).

Electromagnetism 2: Eddy current and decaying magnetic field

An infinitely long, thin-walled cylinder has a circular cross-section with inner radius a and outer radius b. Let the z-axis be coincident with the symmetry axis of the cylinder, and let r be the *cylindrical* radial coordinate, i.e., the distance perpendicular to the z-axis.

The cylinder is made of a non-magnetic metal with electrical conductivity σ. For times $t < 0$ the system has come to equilibrium with an applied external magnetic field. As a result the magnetic field is uniform and constant throughout all space, $\vec{B} = B_c\ \hat{z}$, and the current density is everywhere zero, $\vec{J} = 0$.

At $t = 0$ the sources responsible for the applied magnetic field are turned off. Subsequently, a current develops in the metal which supports a residual magnetic field $\vec{B}(r, t)$ in the z-direction. The residual magnetic field inside the cylinder begins with the constant value B_c and eventually relaxes to zero.

In what follows you should assume the wall of the cylinder is sufficiently thin that the fields and currents do not vary significantly between $r = a$ and $r = b$.

(a) (*3 pts*) A zeroth-order approximation assumes that the residual magnetic field is independent of position inside the cylinder, and zero elsewhere:

$$\vec{B}_0 = \begin{cases} B_0(t)\hat{z} & r \leq a \\ 0 & r > a \end{cases} \tag{1}$$

Derive a differential equation for $B_0(t)$ under these assumptions, and give the relevant solution.

(b) (*4 pts*) Calculate the first r-dependent correction $\vec{B}_1 = B_1(r, t)\hat{z}$ to the residual magnetic field, by finding the electric field $\vec{E}(r, t)$ that is induced by $\vec{B}_0(t)$, and then the magnetic field that is induced by $\vec{E}(r, t)$. You may assume that the correction $\vec{B}_1(r, t)$ vanishes at $r = 0$. You may also express $\vec{B}_1$ in terms of $\vec{B}_0$.

(c) (*3 pts*) What dimensionless parameter must be $\ll 1$ in order to ensure that $\vec{B}_1$ is a small correction? Express this inequality in terms of the time required for light to cross the cylinder.

SECTION III: STATISTICAL MECHANICS

Statistical Mechanics 1: Cooling by demagnetization

Consider an insulating crystal formed by atoms of mass m. Each atom carries a nuclear spin of $\hbar/2$ but no net electron spin. The number density of the atoms is n. The crystal supports sound waves (phonons) with two transverse and one longitudinal mode, all with the same velocity v.

(a) (*4 pts*) First assume that the nuclear spin is completely polarized by a strong magnetic field. Find the free energy per unit volume $f(T)$ of the crystal, where the temperature of the crystal is T. (Your answer may contain a *dimensionless* integral which you need not evaluate.)

(b) (*4 pts*) Initially, the nuclear spin is completely polarized by a strong magnetic field, and the temperature of the crystal is T. Now we slowly decrease the magnetic field to zero while keeping the system isolated from its surroundings. Find the final temperature T' of the crystal after the demagnetization. Here we assume that there is sufficient coupling between the phonon and the spin degrees of freedom that the system remains in thermal equilibrium throughout the demagnetization process. However, the coupling has a negligible effect on the free energy of the system.

(c) (*2 pts*) Following your analysis of part b), are there circumstances in which the crystal remains magnetized even after the external magnetic field is reduced to zero? Explain your result.

Statistical Mechanics 2: Quasi-particle excitations in a fermionic superfluid

A spin-1/2 Fermi gas with attractive interactions between spin up and spin down fermions forms a superfluid of fermion pairs at low temperatures. At zero temperature, all fermions are paired up, with spin up fermions pairing with spin down fermions. However, at small but non-zero temperatures some of these pairs break up, and unpaired, fermionic quasi-particle excitations exist in the system. We will here calculate their contribution to the thermodynamics of the superfluid.

At low temperatures, much smaller than the pairing gap, we can approximate the energy $\epsilon(k)$ of the spin-1/2 quasi-particle excitations at momentum $\hbar k$ by

$$\epsilon(k) \approx \Delta + \frac{\hbar^2}{2m_0}(k - k_F)^2 \tag{1}$$

where Δ is the pairing gap, $m_0 = m\frac{\Delta}{2E_F}$ the quasi-particle mass that differs from the bare mass m of free fermions, $E_F = \frac{\hbar^2 k_F^2}{2m}$ is the Fermi energy, $k_F = (3\pi^2 n)^{1/3}$ is the Fermi wavevector, $n = N/V$ is the total density of the spin-1/2 Fermi gas, N the total number and V the volume of a cubic box confining the gas. We will assume throughout this problem that $\Delta \ll E_F$, well fulfilled for conventional superconductors or weakly interacting Fermi gases, we will assume low temperatures $k_B T \ll \Delta$ and we will neglect interactions between quasi-particles.

(a) (*5 pts*) Using this approximation for $\epsilon(k)$, find the contribution from these fermionic quasi-particle excitations to the free energy of the gas

$$F_{\text{qp}}(N, V, T) \equiv -k_B T \ln Z_{\text{qp}},$$

where Z_{qp} is the partition function for quasi-particle excitations

$$Z_{\text{qp}} \equiv \sum_{\text{all qp states}} e^{-E_{\text{qp}}/k_B T}. \tag{2}$$

The sum is over all states involving any number of quasi-particle excitations, and E_{qp} is the total energy of each such state.

Write your result in the form

$$F_{\text{qp}}(N, V, T) = aNE_F \left(\frac{k_B T}{E_F}\right)^{\alpha} \left(\frac{\Delta}{E_F}\right)^{\beta} f\left(\frac{k_B T}{\Delta}\right) \tag{3}$$

where you need to find the dimensionless constant a, the exponents α and β, and the function $f(x)$ that describes the dependence of F_{qp} on the ratio of temperature $k_B T$ to superfluid gap Δ that cannot be written as a power law.

Even if you do not succeed in finding a, α, β and the functional form $f(x)$, you can do all subsequent parts by expressing your answers to each of them in terms of these constants and the function f.

Problem continued on next page.

You may find the following integral useful:

$$\int_0^\infty \mathrm{d}x \, x^2 \, e^{-(x-x_0)^2} \approx \sqrt{\pi} x_0^2, \tag{4}$$

for $x_0 \gg 1$.

(b) (*2 pts*) Find the contribution to the entropy $S(N, V, T)$ and to the total energy $E_{\mathrm{tot}}(N, V, T)$ of the Fermi gas due to the quasi-particle excitations. *If you did not find the expression for $f(x)$ above, you may assume $xf'(x) \gg f(x)$ for $x \ll 1$.*

(c) (*3 pts*) In atomic Fermi gases, the superfluid gap Δ can be freely adjusted, independently of N, V and T. Let's assume the gas is initially at temperature T_0. If the gap is *adiabatically* changed from an initial value Δ_0 to a final value Δ_1, at constant N and V, what is the final temperature T_1? Assume throughout the evolution that $k_{\mathrm{B}}T \ll \Delta$. *If you did not find the expression for $f(x)$ above, you may assume $xf'(x) \gg f(x)$ and $xf''(x) \gg f'(x)$ for $x \ll 1$.*

SECTION IV: QUANTUM MECHANICS

Quantum Mechanics 1: A Quantum Spin Chain

Consider a one-dimensional chain of N spin-1/2 particles coupled through the Hamiltonian

$$H = J \sum_{i=1}^{N-1} \vec{S}_i \cdot \vec{S}_{i+1}, \tag{1}$$

where $\vec{S} = (S_x, S_y, S_z)$ are the usual spin operators for a spin-1/2 particle, $J > 0$ is a **positive** constant, and $N \gg 1$.

In a famous 1931 paper, Hans Bethe showed that for this Hamiltonian, the ground state energy per particle $E_{\mathrm{GS}}/N \equiv E_0$ was equal to $-\hbar^2 J(\log 2 - 1/4) \approx -0.433\hbar^2 J$. You will not be required to reproduce this result. Instead, you will determine upper and lower bounds for the ground state energy per particle.

(a) (*2 pts*) If the spin operators are treated as classical spin vectors with $|\vec{S}| = \hbar/2$, what is the ground state spin configuration and what is the ground state energy per particle E_0?

(b) (*5 pts*) Consider the trial wave function

$$|\Psi\rangle = \bigotimes_{i=\mathrm{odd}} |i, i+1\rangle_0 = |1, 2\rangle_0 \otimes |3, 4\rangle_0 \otimes \cdots \otimes |N-1, N\rangle_0 \,, \tag{2}$$

where $|i, j\rangle_0$ is the spin singlet state formed from the spins on sites i and j. Use this state to find an **upper** bound on E_0.

(c) (*3 pts*) Prove the following **lower** bound on E_0:

$$-\frac{3}{4}\hbar^2 J \leq E_0. \tag{3}$$

Quantum Mechanics 2: Anomalous Magnetic Moment of the Electron

The gyromagnetic factor of the electron g determines the relationship between the electron magnetic moment $\vec{\mu}$ and the electron spin $\vec{S}$,

$$\vec{\mu} = g\frac{e}{2m}\vec{S}, \tag{1}$$

where e is the electron charge and m is the electron mass. Famously, the Dirac equation predicts $g = 2$, but in quantum electrodynamics, the electron picks up an anomalous magnetic moment $g = 2(1 + a)$, where the current experimental value is $a = 0.00115965218076(27)$.

One way to experimentally measure a is to allow a beam of electrons to interact with a constant magnetic field $\vec{B} = B\hat{z}$ via the Hamiltonian

$$H = \frac{1}{2m}\left(\vec{p} - e\vec{A}\right)^2 - \vec{\mu}\cdot\vec{B}, \tag{2}$$

where $\vec{A}$ is the vector potential. The electrons are confined to the x–y plane, and you can ignore any electron–electron interactions. The electrons will exhibit cyclotron motion with frequency $\omega = eB/m$, but they will also exhibit spin precession with a slightly different frequency. In this problem, you will show how to use this phenomenon to extract a.

(a) (*2 pts*) Verify the commutation relations

$$[v_x, H] = i\hbar\omega v_y, \qquad [v_y, H] = -i\hbar\omega v_x, \tag{3}$$

where $\vec{v} = (\vec{p} - e\vec{A})/m$ is the gauge-invariant velocity operator. [*Hint*: Because $\vec{v}$ is gauge invariant, you are free to choose any gauge for $\vec{A}$ you wish.]

(b) (*6 pts*) Consider the two expectation values

$$C_1(t) = \langle S_x v_x + S_y v_y \rangle, \qquad C_2(t) = \langle S_x v_y - S_y v_x \rangle. \tag{4}$$

Derive a set of coupled differential equations that describe the time evolution of $C_1(t)$ and $C_2(t)$. In the special case that $a = 0$ (i.e. $g = 2$), verify that $C_1(t)$ and $C_2(t)$ do not change with time.

(c) (*2 pts*) A beam of electrons of velocity $\vec{v}$ is prepared at time $t = 0$ in a spin state with known values of $C_1(0)$ and $C_2(0)$. The beam interacts with a magnetic field $\vec{B} = B\hat{z}$ between $t = 0$ and $t = T$. The expectation value $C_1(T)$ is experimentally measured to be periodic with period $2\pi/\Omega$ (i.e. $C_1(T) = C_1(T + 2\pi/\Omega)$). Use this information to determine the value of a in terms of Ω and other physical parameters.