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DEPARTMENT OF PHYSICS

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DOCTORAL GENERAL EXAMINATION

PART II

September 8, 2000

FIVE HOURS

1. This examination is divided into four sections, Mechanics, Electricity & Magnetism, Statistical Mechanics, and Quantum Mechanics, with two problems in each. It is advisable to carefully read both problems in each section before making your choice. Submit ONLY one problem per section. IF YOU SUBMIT MORE THAN ONE PROBLEM FROM A SECTION, BOTH WILL BE GRADED, AND THE PROBLEM WITH THE LOWER SCORE WILL BE COUNTED.

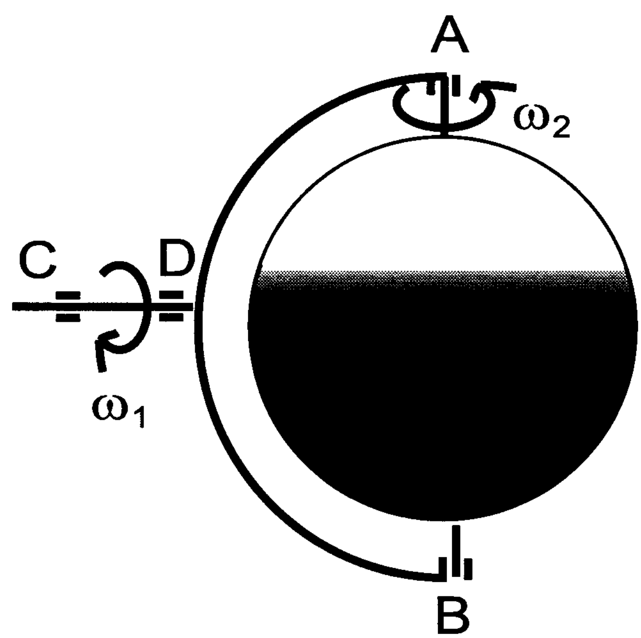
2. Use a separate fold of paper for each problem, and write your name on each fold. Include the problem number with each solution.

3. Calculators may be used.

4. **No books or reference Materials My Be Used.**

Classical Mechanics Problem 1

A spherical container is partially filled with a slowly hardening epoxy resin. The container is spinning with angular velocity ω_2 around axis AB which in turn is spinning with angular velocity ω_1 around the fixed axis CD, with $\omega_1 \neq \omega_2$. The goal of this problem is to find the shape of the surface of the resin after it has hardened. You can assume that the container will make many turns before the resin actually hardens.



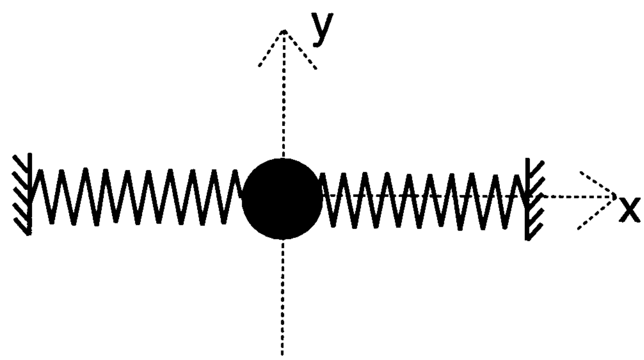
(a) Choose a non-inertial coordinate system moving with the container and find an expression for the instantaneous net force acting on a fluid element. Assume that the resin can still move but it is very viscous, so that Coriolis forces can be ignored.

(b) Find the average force acting on a fluid element in the same system of coordinates.

(c) Obtain the expression for the form of the surface of the resin after it has hardened.

Classical Mechanics Problem 2

A small iron ball of mass m is attached to two identical springs along the x-axis. The spring constants are k and the relaxed lengths are L_0. When aligned along the x-axis the springs are compressed, the distance between their respective points of attachment is $2L < 2L_0$. Initially the ball is constrained to move along the y-axis (see Figure).



(a) Find the potential energy of the ball as a function of displacement y for small amplitude oscillations. Expand it in a power series through terms of order y^4. This power expansion of the potential energy function will suffice for answering parts (b)-(d) of this problem.

A high frequency external force $\vec{F}(t) = -(my_0\omega^2)\cos(\omega t)\hat{y}$ is applied to the iron ball along the y-axis (e.g. by using an external electromagnet). The frequency ω of the external force is much larger than the free oscillation frequency of the ball.

(b) Write equations of motion for the ball. Separate equations of motion into high and low frequency by appropriate substitution of variables.

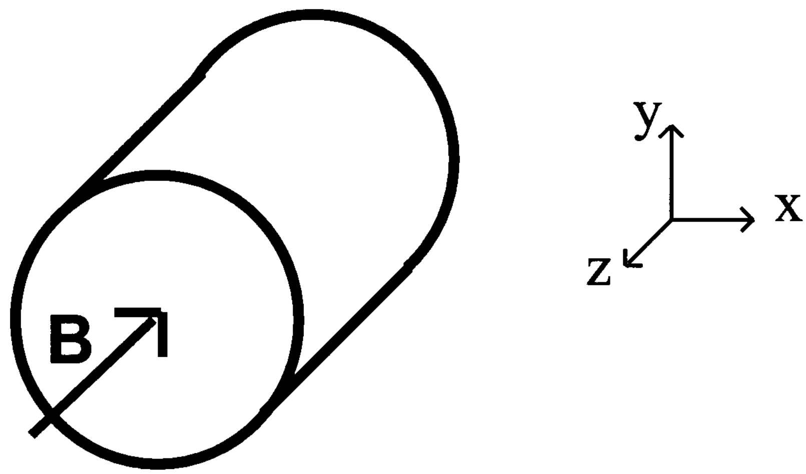
(c) Find the effective potential U_{eff} for the low frequency motion of the ball as a function of given parameters.

(d) Find the equilibrium points and the frequency of small amplitude free (low frequency) oscillations of the ball in the potential U_{eff} as a function of the parameter $T = y_0^2$.

(e) Now assume that the ball is constrained to move along the x-axis and that the external force is also acting along the x-axis, $\vec{F}(t) = -(mx_0\omega^2)\cos(\omega t)\hat{x}$. Find the equilibrium points and the frequency of small amplitude free oscillations in this case.

Electromagnetism Problem 1

Consider a long solid cylinder made of uniform resistive material. The cylinder is in a region in which there is an applied magnetic field that is uniform and is directed along the axis of the cylinder. The magnetic field is time dependent and it is oscillating with angular frequency ω: $\vec{B}(t) = B_z \cos \omega t \;\; \hat{z}$. The length of the cylinder is L and its radius is R ($R \ll L$). The resistivity of the cylinder material is ρ.



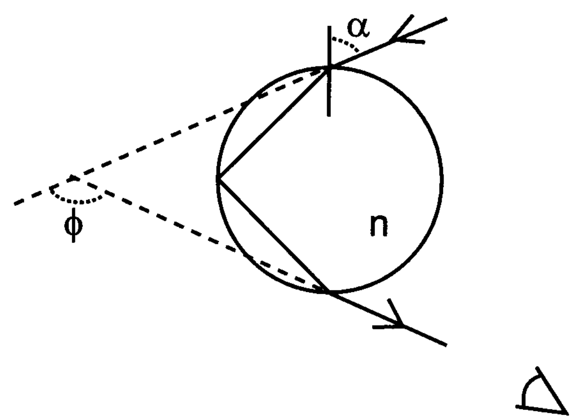
(a) Calculate the current density $\vec{\jmath}(t)$ in the volume of the cylinder. Assume initially that you can ignore the self-inductance of the cylinder. Ignore end effects and the Hall effect.

(b) For large values of ω the effect of self-inductance cannot be ignored. Calculate the correction to the current density $\Delta\vec{\jmath}(t)$ due to the self-inductance of the cylinder in next order of ω.

(c) Give the condition on ω such that the self-inductance of the cylinder can be ignored.

Electromagnetism Problem 2

An airplane has spread a cloud of transparent, liquid insecticide over a large horizontal corn field. The cloud consists of many identical small spherical drops of radius R. The refraction coefficients of the insecticide for the extreme wavelengths of the visual spectrum are $n_{\text{red}} = 1.46$ and $n_{\text{violet}} = 1.47$ for red and violet light respectively. A farmer watching the cloud observed a rainbow. The brightest rainbow occurs when the Sun light rays are deflected towards the observer after a single reflection at the back of the liquid drop. Ignore effects of multiple internal reflections. Assume that the Sun is a point-like source of light very far from the Earth.



(a) Find the total light ray deflection angle $\phi(\alpha)$ as a function of the angle α of light incidence on the drop surface for the coefficient of refraction n (see Figure).

(b) Assume that the drop is illuminated by light of uniform intensity I_0. Find the energy per unit time deflected from a single drop and observed using a small detector of area S (e.g. a human eye) at a large distance $L \gg R$ from the drop as a function of the angles α and $\phi(\alpha)$. Due to multiple crossings of liquid-air boundaries, not all the light will follow the geometrical path sketched in the Figure. For simplicity of your calculations, assume that light losses at the boundaries and inside the liquid do not depend on the incidence angle. Include this cumulative intensity loss as a multiplicative constant factor f in your final answer.

(c) Analyze the formulas obtained in (a) and (b) to find the angular radius $(180° - \phi)$ of the red and violet rings of the rainbow.

Statistical Mechanics Problem 1
One-Dimensional Ideal Gas in a Gravitational Field

You might find some of the following integrals useful for this problem:

$$\int_{-\infty}^{\infty} x^{2n}\, e^{-\frac{1}{2}x^2/\sigma^2}\, dx = (2n-1)!!\,\sqrt{2\pi}\,\sigma^{2n+1}$$

$$\int_{0}^{\infty} x^{2n+1}\, e^{-\frac{1}{2}x^2/\sigma^2}\, dx = 2^n\, n!\,\sigma^{2n+2}\ .$$

where $(2n-1)!! = (2n-1)(2n-3)\ \ldots\ 1$, with $(-1)!! \equiv 1$.

In this problem we consider a classical ideal gas in one dimension. The gas particles, each with mass m, move vertically along a z-axis in a uniform gravitational field of acceleration g, bouncing off a floor located at $z=0$. All interactions between particles are ignored, so the particles pass freely through each other. The Hamiltonian for a single gas particle is

$$H = \frac{q^2}{2m} + V(z)\ ,$$

where

$$q = m\frac{dz}{dt}$$

and

$$V(z) = \begin{cases} mgz & \text{if } z > 0 \\ \infty & \text{if } z < 0\ . \end{cases}$$

(a) Although the interactions of the particles are negligible, we will assume that the particles have nonetheless relaxed to an ideal Maxwell-Boltzmann distribution at temperature T. The distribution $P(q,z)$ is normalized so that $P(q,z)\,dq\,dz$ is the expected number of particles in a phase space box of area $dq\,dz$. If N is the total number of particles in the system, write an explicit expression for the normalized distribution $P(q,z)$.

(b) Now consider a single gas particle, which interacts only with the gravitational field and the floor. Its motion will be periodic, bouncing off the floor and rising to some maximum height h above the floor. Suppose that for a given particle we know the value of h, but we do not know the phase of the motion (i.e., we do not know where along its periodic trajectory the particle is to be found). Write an explicit expression for the phase space probability density $P_1(q,z|h)$ for this one particle, so that $P_1(q,z|h)\,dq\,dz$ represents the probability of finding the particle in a phase space box of area $dq\,dz$. [*Hint:* this expression will involve a Dirac δ-function. Be sure to consider the effect of the particle's speed on the amount of time it spends at any given height.]

— PROBLEM CONTINUES ON NEXT PAGE —

Your answer to (b) should reflect the fact that each gas particle speeds up significantly as it approaches the floor, but it is not obvious how this can be consistent with the Maxwell-Boltzmann distribution. In the next two parts you will be asked to derive the equations needed to relate the Maxwell-Boltzmann distribution to the trajectories of individual particles.

(c) Assuming that the phases of the periodic motions are random, the statistical properties of the gas can be described by specifying the particle height distribution $p(h)$, where $p(h)\,\mathrm{d}h$ is the expected number of particles for which the **maximum** height will be in some interval $\mathrm{d}h$. Given an arbitrary $p(h)$, find the corresponding expression for the full phase space density function $P(q,z)$. To obtain full credit, all integrations must be carried out explicitly.

(d) Assuming again that the gas is described by the Maxwell-Boltzmann distribution, derive an expression for the corresponding distribution $p(h)$, as defined in part (c).

Statistical Mechanics Problem 2
Synthesis of Helium from Protons and Neutrons

You might find some of the following formulas useful for this problem:

$$\int_{-\infty}^{\infty} x^{2n}\, e^{-\frac{1}{2}x^2/\sigma^2}\, \mathrm{d}x = (2n-1)!!\,\sqrt{2\pi}\,\sigma^{2n+1}$$

$$\int_{0}^{\infty} x^{2n+1}\, e^{-\frac{1}{2}x^2/\sigma^2}\, \mathrm{d}x = 2^n\, n!\,\sigma^{2n+2}\ .$$

where $(2n-1)!! = (2n-1)(2n-3)\ \ldots\ 1$, with $(-1)!! \equiv 1$.

In this problem you are asked to investigate the thermal equilibrium of the nuclear reaction

$$n + n + p + p \longleftrightarrow \alpha + \text{photon}\ ,$$

where n and p are the proton and neutron, and α is the alpha particle (nucleus of He^4).

To set the foundations, consider first a system of spinless particles of a single species with mass m. Assuming that we can treat the particles as an ideal noninteracting gas, a grand canonical partition function can be written as

$$Z = \sum_{\substack{\text{all}\\ \text{states}}} e^{-(E-\mu N)/kT}\ ,$$

where E is the total energy of each state, N is the total number of particles in the state, μ is the chemical potential, T is the temperature, and k is Boltzmann's constant. The sum can be calculated for the states in a cube of volume V, with periodic boundary conditions. If we assume that the volume is large, then the sum over allowed single-particle momenta can be replaced by an integral, and the partition function can be written in the form

$$\ln Z = V \int \mathrm{d}^3p\, f(\vec{p}, m, \mu, T)\ .$$

(a) Find the function $f(\vec{p}, m, \mu, T)$ appearing in the above expression for each of three cases:
 (i) a Bose gas
 (ii) a Fermi gas
 (iii) a classical gas, for which the probability of two particles occupying the same quantum state is negligibly small, and for which the energy can be treated nonrelativistically.

— PROBLEM CONTINUES ON NEXT PAGE —

(b) Focussing on the case (iii) above, find an explicit expression for the number density n of particles as a function of m, μ, T, and fundamental constants.

(c) Again assuming that case (iii) prevails, consider the helium synthesis reaction and derive a thermal equilibrium relation of the form

$$\frac{n_n^2\, n_p^2}{n_\alpha} = F(T) \ ,$$

where n_n, n_p, and n_α refer to the number densities of free neutrons, free protons, and α particles, respectively. The function $F(T)$ will depend on the binding energy B of the α particles, and also on the particle masses and fundamental constants. [*Hint:* Remember that the neutron and proton are spin-$\frac{1}{2}$ particles, while the α particle is spinless. If you are not familiar with chemical potential techniques, you should still be able to derive this formula from first principles. Consider a grand canonical partition function for the combined system of protons, neutrons, and α particles, ignoring photons. Assume that the spectrum of states can be approximated by free particles of these three species. Introduce chemical potentials for each of the two conserved quantities, $N_p + 2N_\alpha$ and $N_n + 2N_\alpha$, where N_p, N_n, and N_α denote the number of free protons, free neutrons, and α particles, respectively.]

Quantum Mechanics Problem 1
Angular Momentum and Mixed States

You might find some of the following formulas useful for this problem:

$$J_\pm = J_x \pm iJ_y$$

$$J_\pm \left|J, M\right\rangle = \sqrt{J(J+1) - M(M \pm 1)}\,\left|J, M \pm 1\right\rangle$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

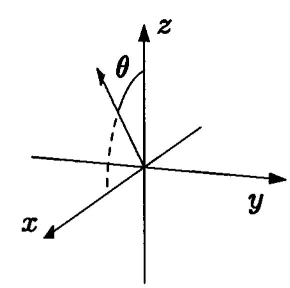
$$R_y(\theta) \equiv e^{-\frac{1}{2}i\sigma_y} = \cos \tfrac{1}{2}\theta \; - i\sigma_y \sin \tfrac{1}{2}\theta$$

$$\cos \tfrac{1}{2}\theta = \sqrt{\frac{1+\cos\theta}{2}} \qquad \sin \tfrac{1}{2}\theta = \sqrt{\frac{1-\cos\theta}{2}}\ .$$

(a) Consider a system consisting of three nonrelativistic distinguishable spin-$\frac{1}{2}$ particles, which will be labeled A, B, and C. The particles are in a state $|\Psi\rangle$ in which particles A and B have a total spin $S_{AB} = 1$ (i.e., $(\vec{s}_A + \vec{s}_B)^2 = S(S+1)\hbar^2$, with $S = 1$), and the total spin of the 3-particle system is $S_{\rm tot} = 1/2$, $M_{\rm tot} = 1/2$ (i.e., $s_{A,z} + s_{B,z} + s_{C,z} = \frac{1}{2}\hbar$). Suppose that one measures the z-component of the spin of particle A. What is the probability that the spin will be measured as up (i.e., $s_{A,z} = +\frac{1}{2}\hbar$)? [*Hint:* to construct the state with $S_{\rm tot} = 1/2$ and $M_{\rm tot} = 1/2$, you may find it useful to first construct the state with $S_{\rm tot} = 3/2$ and $M_{\rm tot} = 1/2$.]

(b) Starting with the same state $|\Psi\rangle$ described in part (a), suppose that we measure the x-component of the spin of particle B. (Note that this measurement is made *instead* of the measurement described in part (a).) What is the probability that this spin is measured as up (i.e., $s_{B,x} = +\frac{1}{2}\hbar$)? Your answer should be derived or explained, not just stated.

(c) Again starting with the same state $|\Psi\rangle$ described in part (a), suppose that we measure the spin of particle C along an axis in the x-z plane, at an angle θ from the positive z-axis in the direction of the positive x-axis (as shown in the diagram). What is the probability that this spin is measured as up?



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(d) Now consider a single spin-$\frac{1}{2}$ particle, for which the spin states can be described in the basis $|j\,m\rangle$, where $j = 1/2$ and $m = +1/2$ or $-1/2$, where $J_z = m\hbar$. In this part you are asked to consider two recipes for producing "mixed states" — that is, states described not by a unique quantum state vector, but instead by a probability distribution of quantum state vectors. The two mixed states are:

(i) a state which has probability 1/2 of being described by either $|j, m{=}{+}\frac{1}{2}\rangle$ or $|j, m{=}{-}\frac{1}{2}\rangle$.

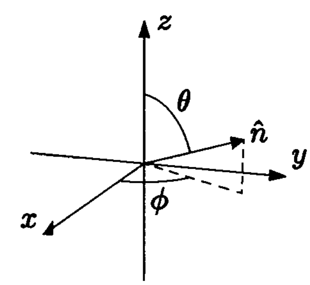
(ii) a state which is constructed from the state vectors

$$|\Psi(\theta,\phi)\rangle = \cos\tfrac{1}{2}\theta\, e^{\frac{1}{2}i\phi}\, |j, m{=}{+}\tfrac{1}{2}\rangle + \sin\tfrac{1}{2}\theta\, e^{-\frac{1}{2}i\phi}\, |j, m{=}{-}\tfrac{1}{2}\rangle \ ,$$

where θ and ϕ have a uniform probability density per solid angle. Note that $|\Psi(\theta,\phi)\rangle$ has been chosen so that

$$\vec{s}\cdot\hat{n}\,|\Psi(\theta,\phi)\rangle = \tfrac{1}{2}\hbar|\Psi(\theta,\phi)\rangle \ ,$$

where $\vec{s}$ is the spin operator and $\hat{n}$ is the unit vector in the (θ,ϕ) direction, where θ and ϕ are the polar and azimuthal angles, respectively, as shown in the diagram.



Uniform probability density per solid angle means that

$$p(\theta,\phi)\,\mathrm{d}\theta\,\mathrm{d}\phi = \frac{1}{4\pi}\sin\theta\,\mathrm{d}\theta\,d\phi \ .$$

Thus, this mixture can be described as being equally likely to have its spin up in any direction.

Is it possible to perform a measurement that distinguishes between these states, or will they give equivalent results for all measurements? If you believe that they are distinguishable, describe a measurement that would give a different outcome for the two cases. If you believe that they are equivalent, explain why.

Quantum Mechanics Problem 2
Time-Dependent Perturbation Theory

You might find the following formula useful for this problem:

$$a^{\dagger n} |0\rangle = \sqrt{n!}\, |n\rangle$$

Consider a particle which for $t < 0$ evolves in one dimension in a quadratic potential, so the Hamiltonian is given by

$$H_0 = \frac{p^2}{2m} + \tfrac{1}{2} k x^2 \ .$$

The particle is initially in the ground state of this Hamiltonian. For $t > 0$ a small perturbation is introduced, so the full Hamiltonian becomes

$$H = H_0 + H_1(t) \ ,$$

where

$$H_1(t) = \epsilon x^2 e^{-\lambda t} \ .$$

(a) Show how to construct creation and annihilation operators $a^\dagger$ and a, as linear combinations of p and x, with the property that

$$[a, a^\dagger] = 1$$

and

$$H_0 = \hbar\omega \left(a^\dagger a + \tfrac{1}{2}\right) \ ,$$

where $\omega = \sqrt{k/m}$.

(b) Express the perturbation Hamiltonian $H_1(t)$ in terms of creation and annihilation operators.

(c) To the lowest nonvanishing order in ϵ, find the probability that at late times ($t \gg 1/\lambda$) the particle will be found in the 2nd excited state of H_0.

(d) To the lowest nonvanishing order in ϵ, extend the calculation in (c) to find the probability that the particle will be found in the nth excited state of H_0, for all values of n.