MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF PHYSICS

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DOCTORAL GENERAL EXAMINATION

PART II

August 31, 2001

FIVE HOURS

1. This examination is divided into four sections, Mechanics, Electricity & Magnetism, Statistical Mechanics, and Quantum Mechanics, with two problems in each. Read both problems in each section carefully before making your choice. Submit ONLY one problem per section. IF YOU SUBMIT MORE THAN ONE PROBLEM FROM A SECTION, BOTH WILL BE GRADED, AND THE PROBLEM WITH THE LOWER SCORE WILL BE COUNTED.

2. Use a separate paper (i.e. folded sheet) for each problem. Write your name and the problem number (I.2 for example) on each folded sheet.

3. Calculators may be used.

4. No books or reference materials may be used.

CLASSICAL MECHANICS: BINARY SCATTERING

Two point masses M_1 and M_2 are in a bound Keplerian orbit when a third body of mass M_3 makes a distant passage. Throughout this problem, "binary" refers to the pair of bodies of masses M_1 and M_2. The third body interacts gravitationally with the binary but is gravitationally unbound.

a) Let the center-of-mass position and separation of the binary be $\vec{R}$ and $\vec{r}$, respectively, and let $\vec{R}+\vec{s}$ be the position of the third body. Starting from the Lagrangian and assuming $r \ll s$, derive the Hamiltonian for the 3-body system, $H = H_r + H_{Rs}$ where

$$H_r = \frac{p_r^2}{2\mu} - \frac{GM_1M_2}{r}\ , \quad H_{Rs} = \frac{1}{2}\frac{|\vec{p}_R - \vec{p}_s|^2}{(M_1+M_2)} + \frac{p_s^2}{2M_3} - \frac{G(M_1+M_2)M_3}{s}$$

with $\mu = M_1M_2/(M_1+M_2)$, and show that the neglected terms are $O(r/s)^2$ times the last term.

b) The Kepler solution for a binary orbit may be written

$$r(\phi) = \frac{a(1-e^2)}{1+e\cos\phi}\ .$$

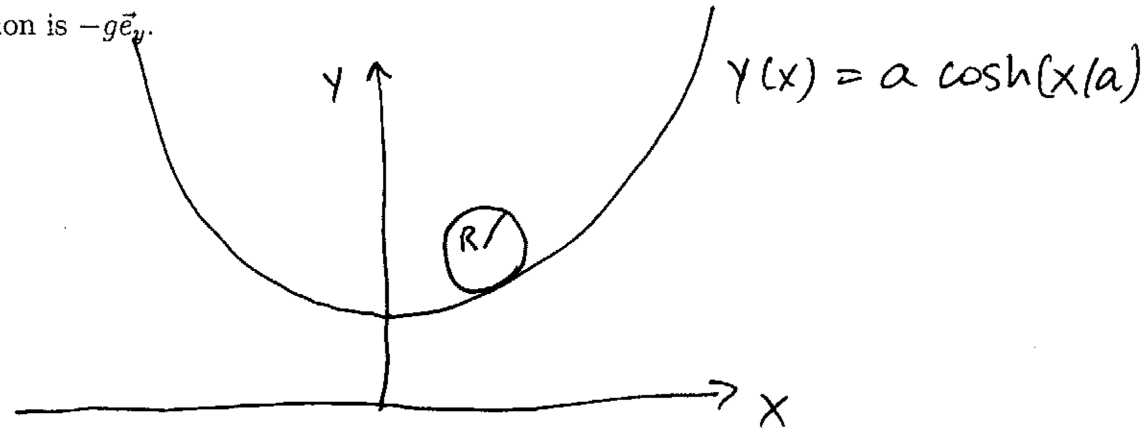
Express a and e in terms of the energy $E<0$ and angular momentum L of the binary consisting of M_1 and M_2 as well as any other parameters needed.

c) Since $\partial H/\partial\vec{R} = 0$, it is easy to solve for the motion of the remaining coordinate degrees of freedom associated with $\vec{R}$ and $\vec{s}$. Working in a frame where the total momentum of the three bodies vanishes, show using Hamilton's equations that $\vec{s}$ also obeys the Kepler solution. Express the resulting orbital parameters a_s and e_s in terms of the energy E_s and angular momentum L_s of the third body.

d) Suppose that the third body has initial speed at infinity v_∞ relative to the center of mass of the binary. After it passes closes to the binary and returns to infinity, what is the recoil speed of the binary, measured in the initial rest frame of the binary? Express your answer in terms of v_∞, e_s, and the masses of the bodies.

CLASSICAL MECHANICS: ROLLING DISK

A uniform circular disk of radius R and mass M rolls in the x-y plane without slipping on the ramp whose shape is given by $y(x) = a\cosh(x/a)$ where $a > R$. The gravitational acceleration is $-g\vec{e}_y$.



a) Let ϕ denote the rotation angle of the disk, chosen so that $\phi = 0$ when the center of the disk is at $x = 0$. Show that the center of the disk has coordinates

$$x = a\sinh^{-1}(R\phi/a) - \frac{R^2\phi}{\sqrt{a^2 + R^2\phi^2}}\,, \quad y = \sqrt{a^2 + R^2\phi^2} + \frac{aR}{\sqrt{a^2 + R^2\phi^2}}\,.$$

b) Find the Lagrangian $L(\phi, \dot{\phi})$ describing the system, assuming that the disk rolls without slipping.

c) Find the angular frequency of small oscillations when the disk is near the bottom of the potential, $\phi^2 \ll (a/R)^2$. Your answer should depend on g, R, and a.

d) Find the normal and tangential contact forces applied by the ramp on the disk when it is released from rest ($\dot{\phi} = 0$), in terms of mg, a, R, and ϕ. You may assume $\phi^2 \ll (a/R)^2$.

e) The coefficient of static friction is $\mu = 0.5$. Will the disk slip initially when it is released from $0 < \phi \ll a/R$?

ELECTROMAGNETISM: ELECTROMAGNETIC WAVES

The Maxwell equations may be written in the covariant form $\nabla_\nu F^{\mu\nu} = -4\pi J^\mu$ together with $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ where $A_\mu(x)$ is the vector potential. Greek indices range over $\{t, x, y, z\}$ and repeated indices are summed over. We assume flat spacetime with Cartesian coordinates so that $\nabla_\mu = \partial/\partial x^\mu$. The Minkowski metric $\eta_{\mu\nu} = \eta^{\mu\nu} = \text{diag}(+1, -1, -1, -1)$ is used to raise and lower indices: $A^\mu = \eta^{\mu\nu} A_\nu$, $A_\mu = \eta_{\mu\nu} A^\nu$. We choose units so that $c = 1$.

a) Starting from the Maxwell equations, and imposing a gauge condition on A^μ, derive the wave equation

$$\Box A^\mu \equiv \eta^{\alpha\beta}\partial_\alpha\partial_\beta A^\mu = 4\pi J^\mu \ .$$

What gauge condition is needed to obtain this result? Show that the A^μ so determined is not unique.

b) The radiative solution of the wave equation is

$$A^\mu(x) = \int \frac{J^\mu(\underline{x}', t_r)}{|\underline{x} - \underline{x}'|}\, d^3x' \ ,$$

where $t_r = t - |\underline{x} - \underline{x}'|$ is the retarded time. Using this, derive the Liénard-Wiechert potentials for a point charge q with trajectory $\underline{x}_q(t)$,

$$A^\mu(\underline{x}, t) = \left[\frac{q(1, \underline{v}_q)}{R - \underline{R}\cdot\underline{v}_q}\right]_{\text{ret}}$$

where $\underline{R} = \underline{x} - \underline{x}_q$ and where $\underline{x}_q$ and $\underline{v}_q = d\underline{x}_q/dt$ are evaluated at the retarded time $t_r = t - R$.

c) A relativistic electron undergoes circular motion in a uniform magnetic field $\underline{B} = B\underline{e}_z$. Find the radius a of the circle and the angular frequency ω of the orbit ($\omega = 2\pi/\text{Period}$) in terms of the electron momentum p and any other needed quantities.

d) Find all four components $A^\mu(z, t)$ for $x = y = 0$ produced by the electron circling in the x-y plane. The center of the electron's orbit is $x = y = z = 0$. Your answers should depend on e, p, m, ω, a, z, and t.

ELECTROMAGNETISM: MODES IN A WAVEGUIDE

a) Consider electromagnetic waves in a waveguide of rectangular cross-section $a \times b$. There are two kinds of waves: TE and TM. The TE (TM) modes have the electric (magnetic) field transverse to $\vec{k}$, respectively, where $\vec{k}$ points down the waveguide. Show that:

(i) The TE modes are given by the solutions of the boundary value problem for the magnetic field component parallel to the wavevector, $[\nabla_\perp^2 + (\omega^2/c^2 - k^2)]\, B_\parallel = 0$, with boundary condition $\partial B_\parallel / \partial n = 0$.

(ii) The TM modes are given by the solutions of the boundary value problem for the electric field component parallel to the wavevector, $[\nabla_\perp^2 + (\omega^2/c^2 - k^2)]\, E_\parallel = 0$, with boundary condition $E_\parallel = 0$.

b) Find the TE and TM modes for the waveguide of part a). That is, find $B_\parallel$ and $E_\parallel$ for cases (i) and (ii) respectively, and find the dispersion relations $\omega(k)$ for all modes. Give expressions for the cutoff frequencies.

c) Consider electromagnetic waves within the space between two perfectly conducting planes with the wavevector $\vec{k}$ parallel to the planes. The distance between the planes is a.

Find all electromagnetic modes. Show that, in addition to the TE and TM modes, in this case there is also a so-called TEM mode in which both the electric and magnetic fields are transverse to $\vec{k}$. Show that the TEM mode is give by the solution to the equation $\nabla_\perp^2 \Phi = 0$ for the electric potential and that dispersion relation for the TEM mode is $\omega = kc$.

d) Consider the modes found in part b) for a rectangular waveguide and take the limit $b \to \infty$. Which of the modes becomes the TEM mode?

Quantum Mechanics I : Hydrogen-like Atoms

The Schroedinger equation for an electron of mass m in a spherically symmetric potential $V(r)$ is conveniently written as

$$\left[-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right)+\frac{1}{2mr^2}\vec{L}^2+V(r)\right]\psi(\vec{r})=E\psi(\vec{r})$$

where $\vec{L}$ is the angular momentum operator. Throughout we ignore the spin of the electron.

a) Consider the hydrogen atom with the potential:

$$V(r)=-\frac{e^2}{r}.$$

Assume that ψ is an s-wave so that we have

$$\psi(\vec{r})=NR(r)$$

for some function $R(r)$ with N being a normalization constant.
Assume $R(r)$ is of the form $e^{-\beta r}$. Find the energy eigenvalue E and also β in terms of $\hbar, m$ and e.

b) An electron is in the ground state of tritium which has a nucleus with one proton and two neutrons. A nuclear reaction instantaneously changes the nucleus to He^3 which has two protons and one neutron. Calculate the probability that the electron is in the ground state of He^3 after the transition. (Useful integral $\int_0^\infty r^2\exp(-\gamma r)dr=2/\gamma^3$.)

c) An electron in a hydrogen atom has orbital angular momentum one, so its wave function is of the form

$$\psi(\vec{r})=\widetilde{R}(r)Y^1_m(\theta,\phi)$$

for some function $\widetilde{R}(r)$ (please note that m above is a quantum number not a mass). Assume that $\widetilde{R}$ is of the form $r\exp(-\tilde{\beta}r)$. Find the energy eigenvalue and also $\tilde{\beta}$ in terms of $\hbar, m$ and e.

Quantum Mechanics II : Particle in a well

a) Consider a quantum system with a hamiltonian H with eigenstates $|a\rangle$ satisfying

$$H|a\rangle = E_a|a\rangle, \quad \text{and} \quad E_0 \leq E_1 \leq E_2 \leq \cdots$$

Prove that for any normalized state $|\psi\rangle$

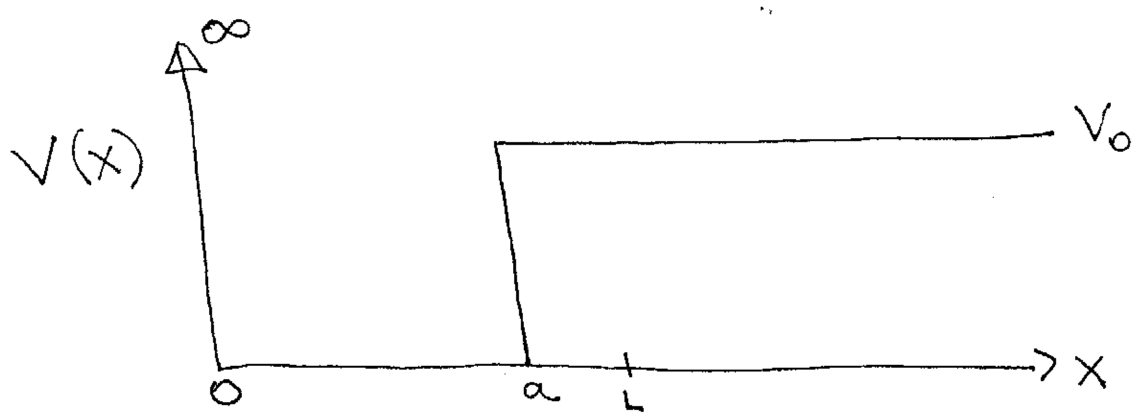
$$E_0 \leq \langle\psi|H|\psi\rangle .$$

b) We now consider a variational estimate of the ground state energy for a particle of mass m moving in the potential

$$V(x) = \begin{cases} \infty, & x < 0 \\ 0, & 0 < x < a \\ V_0, & a < x . \end{cases}$$

The hamiltonian is

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x) .$$



As a variational ansatz use $\psi(x) \sim \sin(\pi x/L)$, for $0 \leq x \leq L$, and $\psi(x) = 0$ otherwise, where the variational parameter L is slightly larger than a. Here you will assume that $V_0 >> \frac{\hbar^2}{ma^2}$, namely, the height of the potential is much larger than the ground state energy. Calculate an approximate value for

$$\Delta \equiv L - a << a$$

by minimizing the expectation of the hamiltonian in the state $|\psi\rangle$.

c) Calculate to leading order in Δ/a the ground state energy.

STAT MECH I

PHYSICAL ADSORPTION

A classical gas of non-interacting atoms is in thermal equilibrium at temperature T in a container of volume V and surface area A. The potential energy of the atoms in the bulk is zero. Atoms adsorbed on the surface have a potential energy $V = -E_0$ and behave as an ideal two-dimensional gas.

Find an analytic expression for the surface density $\sigma(n, T) \equiv N_{surface}/A$ in terms of the bulk density $n \equiv N_{bulk}/V$ and the temperature. Be sure to use "correct Boltzmann counting". The correct expression will show the interesting result that σ would be zero if $\hbar$ were not finite.

The following mathematical results may be useful.

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-x^2/2)\, dx = 1 \qquad N! \approx (N/e)^N$$

STAT MECH II

BOSE-EINSTEIN CONDENSATION

Consider a gas of N non-interacting spin-S Bose particles in a box of volume V in thermal equilibrium at temperature T. As the temperature is lowered the chemical potential $\mu(T)$ changes as well. At some critical temperature T_c the chemical potential can no longer change without giving rise to an unphysical result and μ becomes "pinned". For $T < T_c$ the chemical potential remains fixed at that value and the ground single-particle state develops a macroscopic occupation; that is, it contains a temperature dependent fraction of all the N particles.

Using the above description as a guide

a) Find an analytic expression for the critical temperature T_c as a function of the density of the gas $n \equiv N/V$ and the spin S.

b) Find the fraction of atoms in the ground state $N_0(T)/N$ as a function of temperature for $T < T_c$.

You may express your answer in terms of one or more of the following dimensionless integrals.

$$
\begin{aligned}
I_a &= \int_0^\infty \frac{x}{e^x + 1}\, dx \\
I_b &= \int_0^\infty \frac{x^3}{e^x + 1}\, dx \\
I_c &= \int_0^\infty \frac{x^{1/2}}{e^x - 1}\, dx \\
I_d &= \int_0^\infty \frac{x^{3/2}}{e^x - 1}\, dx \\
I_e &= \int_0^\infty \frac{x^3}{e^x - 1}\, dx
\end{aligned}
$$