

Classical Mechanics I: Central Potential

a) Integrals of motion for a central potential $V(r)$:

Angular momentum $L = r v_t = r^2 \dot{\phi}$ ($v_t =$ tangential velocity)

Energy per unit mass $E = \frac{1}{2} (\dot{r}^2 + v_t^2) + V(r) = \frac{1}{2} \dot{r}^2 + V_{\text{eff}}(r)$,

$$V_{\text{eff}}(r) = V(r) + \frac{L^2}{2r^2}$$

Circular orbit: $\dot{r} = 0 \Rightarrow \frac{dV_{\text{eff}}}{dr} = 0 \Rightarrow \frac{dV}{dr} = \frac{L^2}{r^3} = \frac{v_t^2}{r} = r \dot{\phi}^2$

$$\Rightarrow \dot{\phi} = \omega_{\phi} = \frac{L}{r^2} = \left(\frac{1}{r} \frac{dV}{dr} \right)^{1/2}, \quad P_{\phi} = \frac{2\pi}{\omega_{\phi}} = 2\pi \left(\frac{1}{r} \frac{dV}{dr} \right)^{-1/2}$$

b) $r(t) = r_0 + \epsilon(t)$ with $(dV_{\text{eff}}/dr)(r_0) = 0$, $\epsilon^2 \ll r_0^2$.

Energy per unit mass: $E = \frac{1}{2} \dot{\epsilon}^2 + V_{\text{eff}}(r_0 + \epsilon)$.

Taylor expand: $V_{\text{eff}}(r_0 + \epsilon) = V_{\text{eff}}(r_0) + \left. \frac{dV_{\text{eff}}}{dr} \right|_{r_0} \epsilon + \frac{1}{2} \left. \frac{d^2 V_{\text{eff}}}{dr^2} \right|_{r_0} \epsilon^2 + O(\epsilon^3)$

$$\hookrightarrow E - V_{\text{eff}}(r_0) = \frac{1}{2} \dot{\epsilon}^2 + \frac{1}{2} \omega_r^2 \epsilon^2 + O(\epsilon^3) = \text{constant}$$

$$\omega_r = \left(\frac{d^2 V_{\text{eff}}}{dr^2} \right)_{r_0}^{1/2}$$

this is the simple harmonic oscillator equation, and its general solution is

$$\epsilon(t) = \frac{\sqrt{2(E - E_0)}}{\omega_r} \cos[\omega_r(t - t_0)]$$

where $E_0 \equiv V_{\text{eff}}(r_0)$ and t_0 is an arbitrary constant.

Now write ω_r in terms of $V(r)$ instead of $V_{\text{eff}}(r)$.

$$\omega_r^2 = \frac{d^2 V_{\text{eff}}}{dr^2} = \frac{d^2 V}{dr^2} + \frac{3L^2}{r^4} = \frac{d^2 V}{dr^2} + 3\omega_{\phi}^2 = \frac{d^2 V}{dr^2} + \frac{3}{r} \frac{dV}{dr}$$

$$\omega_r = \left[\frac{d^2 V}{dr^2} + \frac{3}{r} \frac{dV}{dr} \right]_{r_0}^{1/2} = \left[\frac{1}{r^3} \frac{d}{dr} \left(r^3 \frac{dV}{dr} \right) \right]_{r_0}^{1/2}$$

Radial period

$$P_r = \frac{2\pi}{\omega_r}$$

Classical Mechanics I - continued

c) Stability is determined by the sign of ω_r^2 : $\omega_r^2 > 0$ for stability.

$$\omega_r^2 = \frac{1}{r^3} \frac{d}{dr} \left(r^3 \frac{dV}{dr} \right), \quad V(r) = -\frac{GM}{r} e^{-kr}$$

$$\Rightarrow \omega_r^2 = \frac{GM}{r^3} e^{-kr} [1 + kr - (kr)^2]$$

$$1 + kr - (kr)^2 = \left(\frac{\sqrt{5}-1}{2} + kr \right) \left(\frac{\sqrt{5}+1}{2} - kr \right) \neq 0 \text{ only if } kr < \frac{\sqrt{5}+1}{2}$$

\therefore The circular orbits are unstable for $kr > \frac{\sqrt{5}+1}{2}$

d) The outermost stable circular orbit is at $r_0 = \frac{\sqrt{5}+1}{2k}$.

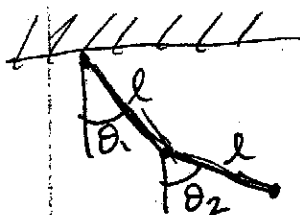
It has energy $E = V(r_0) + \frac{1}{2} (r_0 \omega_r)^2 = V(r_0) + \frac{1}{2} \left(r \frac{dV}{dr} \right)_{r_0}$
per unit mass

$$E = \frac{GM}{r_0} e^{-kr_0} \left[\frac{1}{2} (kr_0 - 1) \right] = \frac{GM}{r_0} e^{-kr_0} \left(\frac{\sqrt{5}-1}{4} \right) > 0$$

If r_0 is decreased slightly, the orbit is absolutely stable and $E > 0$. The effective potential for the Yukawa potential has the form shown.



Classical Mechanics II: Planar Double Pendulum



a) $L = T - V$

For the first rod,

$$T_1 = \frac{1}{2} \int_0^l (r \dot{\theta}_1)^2 \frac{m}{l} dr = \frac{1}{6} m l^2 \dot{\theta}_1^2$$

For the second rod, must apply a velocity offset.
Working in Cartesian coordinates,

$$T_2 = \frac{1}{2} \int_0^l \left[(l \dot{\theta}_1 \cos \theta_1 + r \dot{\theta}_2 \cos \theta_2)^2 + (l \dot{\theta}_1 \sin \theta_1 + r \dot{\theta}_2 \sin \theta_2)^2 \right] \frac{m}{l} dr$$

$$= \frac{1}{2} m l^2 \left[\dot{\theta}_1^2 + \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) + \frac{1}{3} \dot{\theta}_2^2 \right]$$

The potential energies are simply mgz where z is the vertical distance from the ceiling (negative for positions below the ceiling). Since the rods are uniform, we may treat them as if all the mass is concentrated at the center.

$$\therefore V_1 = mg \left[-\frac{l}{2} \cos \theta_1 \right] \quad V_2 = mg \left[-l \cos \theta_1 - \frac{l}{2} \cos \theta_2 \right]$$

Now expand the cosines for small angles: $\cos \theta \approx 1 - \frac{\theta^2}{2}$.

$$\Rightarrow L = T_1 + T_2 - V_1 - V_2 = \boxed{m l^2 \left(\frac{2}{3} \dot{\theta}_1^2 + \frac{1}{2} \dot{\theta}_1 \dot{\theta}_2 + \frac{1}{6} \dot{\theta}_2^2 \right) - m g l \left(\frac{3}{4} \theta_1^2 + \frac{1}{4} \theta_2^2 \right) + \text{constant}}$$

↑
minus sign

Classical Mechanics II - continued

b) Lagrange's equations: $\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_i} \right) - \frac{\partial L}{\partial \theta_i} = 0$

$$\frac{4}{3} \ddot{\theta}_1 + \frac{1}{2} \ddot{\theta}_2 + \frac{g}{l} \left(\frac{3}{2} \theta_1 \right) = 0$$

$$\frac{1}{2} \ddot{\theta}_1 + \frac{1}{3} \ddot{\theta}_2 + \frac{g}{l} \left(\frac{1}{2} \theta_2 \right) = 0$$

Normal modes: $\theta_1 = \hat{\theta}_1 e^{i\omega t}$, $\theta_2 = \hat{\theta}_2 e^{i\omega t}$

$$\underbrace{\begin{pmatrix} \left(\frac{4}{3} \omega^2 - \frac{3g}{2l} \right) & \frac{1}{2} \omega^2 \\ \frac{1}{2} \omega^2 & \left(\frac{1}{3} \omega^2 - \frac{g}{2l} \right) \end{pmatrix}}_M \begin{pmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \end{pmatrix} = 0$$

Let $\frac{g}{l} \equiv \omega_0^2$, $x \equiv \frac{\omega^2}{\omega_0^2}$

$$\Rightarrow \det M = 0 \Rightarrow \left(\frac{4}{3}x - \frac{3}{2} \right) \left(\frac{1}{3}x - \frac{1}{2} \right) - \frac{x^2}{4} = 0$$

$$\frac{7}{36} x^2 - \frac{7}{6} x + \frac{3}{4} = 0 \Rightarrow x = \boxed{3 \pm \frac{6}{\sqrt{7}} = \frac{\omega^2 l}{g}}$$

Normal mode frequency

c) Find eigenvectors.

First case: $\omega^2 = \left(3 - \frac{6}{\sqrt{7}} \right) \frac{g}{l}$ (low frequency mode)

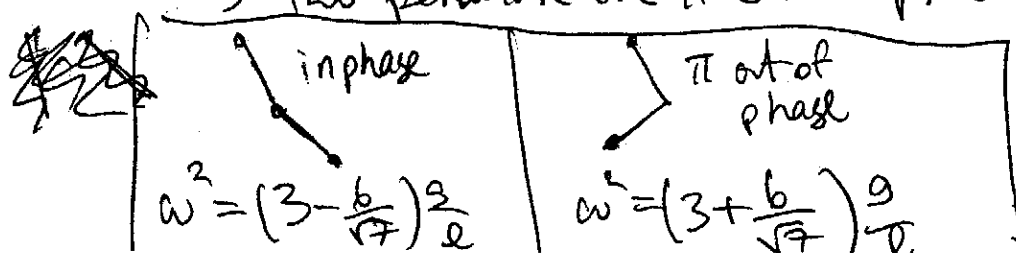
$$\hat{\theta}_2 = \left(\frac{3}{x} - \frac{8}{3} \right) \hat{\theta}_1 = \frac{1}{3} (2\sqrt{7} - 1) \hat{\theta}_1$$

$\frac{1}{3} (2\sqrt{7} - 1) > 0$ and real \Rightarrow the two pendula are in phase.

Second case: $\omega^2 = \left(3 + \frac{6}{\sqrt{7}} \right) \frac{g}{l}$ (high-frequency mode)

$$\hat{\theta}_2 = \left(\frac{3}{x} - \frac{8}{3} \right) \hat{\theta}_1 = \frac{1}{3} (-2\sqrt{7} - 1) \hat{\theta}_1; \quad \frac{1}{3} (-2\sqrt{7} - 1) < 0, \text{ real}$$

\Rightarrow two pendula are π out of phase.



Problem 1 E and M

a) Normal modes are products of harmonic standing waves in the x , y and z directions:

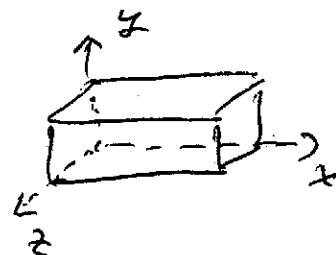
$$\omega = c \sqrt{k_x^2 + k_y^2 + k_z^2} = c \left[\left(\frac{\pi n_x}{a} \right)^2 + \left(\frac{\pi n_y}{b} \right)^2 + \left(\frac{\pi n_z}{b} \right)^2 \right]^{1/2}$$

$$n_{x,y,z} \in \mathbb{Z}_+$$

The lowest frequency has $n_x=1$, $n_{y,z}=0$

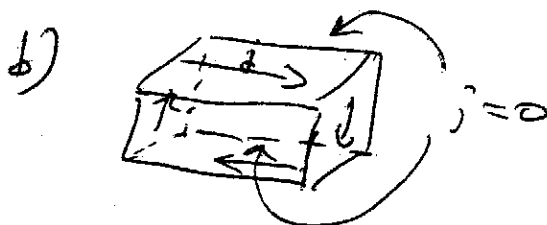
$$\omega = \frac{\pi c}{a}$$

$$\begin{cases} E(\mathbf{r}, t) = E_0 \sin kx e^{-i\omega t} \hat{y} \\ B(\mathbf{r}, t) = B_0 \cos kx e^{-i\omega t} \hat{z} \end{cases}$$



Satisfies Maxwell eqs.

when $\underline{E_0 = i B_0}$

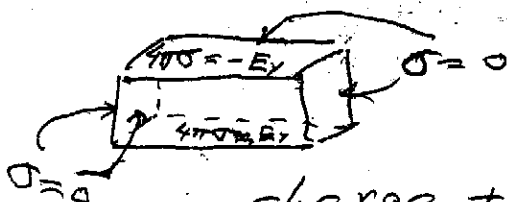


(top/bottom)

$$\frac{4\pi}{c} \int_x j_z = \mp B_z = \mp B_0 \cos kx e^{-i\omega t}$$

$$\frac{4\pi}{c} \int_y j_x = -B_z(x=0) = -B_0 e^{-i\omega t}$$

$$\frac{4\pi}{c} \int_y j_x = B_z(x=a) = -B_0 e^{-i\omega t}$$



charge $\neq 0$ on the top and bottom surfaces only

$$4\pi\sigma = \mp E_y = \mp E_0 \sin kx e^{-i\omega t}$$

c) The energy $W = \iiint dx dy dz \left\langle \frac{E^2}{8\pi} + \frac{B^2}{8\pi} \right\rangle_{\text{time-ave}}$
 $= \underbrace{ab^2}_{\text{Volume}} \frac{E_0^2}{8\pi} \underbrace{\frac{1}{2} \times \frac{1}{2}}_{\text{time-ave}} + ab^2 \frac{B_0^2}{8\pi} \underbrace{\frac{1}{2} \times \frac{1}{2}}_{\text{space-ave}}$

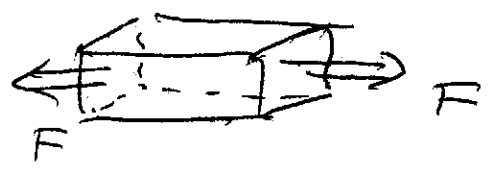
$E_0^2 = 16\pi W / ab^2$

$|B_0| = |E_0|$

The force on the sides $b \times b$ is purely magnetic ($\sigma = 0$)

$F = b^2 \frac{1}{c} j B \times \frac{1}{2} = \frac{b^3}{2c} \frac{c}{4\pi} B_0^2 \frac{1}{2} = \frac{b^3}{16\pi} B_0^2$
 effective field on the current

$F = \frac{W}{a}$ (outward direction)



d) On the top side:

Electric force $F_E = ab \frac{1}{2} \langle \sigma E \rangle = \frac{ab}{2} \frac{E_0^2}{4\pi} \frac{1}{2} R_{\text{time-ave}}$
 (inward direction) effective field on the charge

Magnetic force $F_M = ab \frac{1}{2} \frac{1}{c} \langle j B \rangle = \frac{ab}{2} \frac{B_0^2}{4\pi} \frac{1}{2}$
 (outward direction) effective field time-ave

$F_E + F_M = 0 \rightarrow$ no net force
on the top and bottom
sides.

e) From Maxwell stress tensor, the force per unit surface area is

$$\vec{F} = \frac{1}{4\pi} \vec{E} (\vec{E} \cdot \vec{n}) - \frac{E^2}{8\pi} \vec{n} + \frac{1}{4\pi} \vec{B} (\vec{B} \cdot \vec{n}) - \frac{B^2}{8\pi} \vec{n}$$

where \vec{n} is the normal vector

(i) on the bxb sides:

$$E = 0, B \perp n \rightarrow F = -\frac{B^2}{8\pi} \vec{n}$$

$$F_{\text{t-ave}} = \frac{B_0^2}{8\pi} \frac{1}{2} = \underline{\underline{\frac{B_0^2}{16\pi}}} \text{ (per unit area)}$$

(ii) on the top side:

$$E \parallel n, B \perp n \rightarrow F = \left(\frac{E^2}{8\pi} - \frac{B^2}{8\pi} \right) \vec{n}$$

$$\text{However, } E_0 = B_0 \rightarrow \underline{\underline{F = 0}}$$

Agrees w. e), d)

EM and M

Electromagnetism I: EM Waves in a dilute gas
(see Feynman lectures on Physics, vol. II, Chapter 32)

- a) EM wave travelling along \hat{x} ; assume $\vec{E} \parallel \vec{e}_y$.
Then the electron in the atom behaves classically as a damped, driven harmonic oscillator,

$$m_e (\ddot{y} + \gamma \dot{y} + \omega_0^2 y) = -eE, \quad E = E_0 e^{-i\omega t}$$

$$\Rightarrow y(t) = \frac{e/m_e}{\omega^2 - \omega_0^2 + i\gamma\omega} E$$

\Rightarrow dipole moment per unit volume

$$P = N_a (-e)y = \frac{N_a e^2/m_e}{\omega_0^2 - \omega^2 - i\gamma\omega} E$$

$$\Rightarrow \alpha(\omega) \equiv \frac{P}{\epsilon_0 E} = \frac{N_a e^2}{\epsilon_0 m_e} \frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega}$$

A quantum mechanical derivation would give this same expression multiplied by the oscillator strength f for the transition.

- b) Maxwell eqs. $\nabla \cdot \underline{D} = 0, \nabla \cdot \underline{B} = 0, \nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}, \nabla \times \underline{H} = \frac{\partial \underline{D}}{\partial t}$
with no free charges or currents.

$$\underline{B} = \mu_0 \underline{H}, \quad \underline{D} = \epsilon_0 \underline{E} + \underline{P} = \epsilon_0 (1 + \alpha) \underline{E} \text{ for a single freq. } \omega$$

$$\Rightarrow \frac{\partial^2 \underline{D}}{\partial t^2} - \frac{1}{\mu_0 \epsilon_0 (1 + \alpha)} \nabla^2 \underline{D} = 0, \quad \text{now let } \underline{D} \propto e^{i(kx - \omega t)}$$

$$\Rightarrow k^2 = \mu_0 \epsilon_0 (1 + \alpha) \omega^2 = (1 + \alpha) \frac{\omega^2}{c^2} = \frac{\omega^2 n^2}{c^2}, \quad \boxed{n(\omega) = \sqrt{1 + \alpha(\omega)}}$$

One can also get this result using the microscopic

$$\underline{E}, \underline{B}, \text{ and } \underline{P} \text{ fields: } \nabla \cdot \underline{E} = -(\nabla \cdot \underline{P})/\epsilon_0, \quad c^2 \nabla \times \underline{B} = \frac{\partial}{\partial t} (\underline{P} + \epsilon_0 \underline{E})$$

$$\Rightarrow \frac{\partial^2 \underline{E}}{\partial t^2} - c^2 \nabla^2 \underline{E} = -\frac{1}{\epsilon_0} \frac{\partial^2 \underline{P}}{\partial t^2}; \quad \text{also } \frac{\partial^2 \underline{P}}{\partial t^2} + \gamma \frac{\partial \underline{P}}{\partial t} + \omega_0^2 \underline{P} = \frac{N_a e^2}{m_e} \underline{E}$$

Together these $\Rightarrow k^2 = (1 + \alpha) \frac{\omega^2}{c^2}$, as before.

for a plane wave (Note: we're neglecting dipole-dipole interactions in the dilute gas.)

Electromagnetism I - continued

c) Fourier analysis \rightarrow

$$E(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \exp[ikx - i\omega(k)t] \hat{E}(k) \text{ where}$$

$$\hat{E}(k) = \int_{-\infty}^{\infty} dx e^{-ikx} E(x,0)$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} dx e^{-i(k-k_c)x - x^2/2\sigma^2} = e^{-(k-k_c)^2\sigma^2/2}$$

Now Taylor expand $\omega(k)$ about $k=k_c$:

$$\omega(k) = \omega(k_c) + \left(\frac{d\omega}{dk}\right)_c (k-k_c) + O(k-k_c)^2$$

$$\equiv k_c v_{ph} + v_g (k-k_c) + O(k-k_c)^2$$

$$v_{ph} \equiv \frac{\omega}{k} = \text{phase velocity}, \quad v_g \equiv \frac{d\omega}{dk} = \text{group velocity.}$$

Let $K \equiv k - k_c$. Then

$$E(x,t) = \int_{-\infty}^{\infty} \frac{dK}{2\pi} \exp\left[ik_c(x - v_{ph}t) + iK(x - v_g t) - \frac{K^2\sigma^2}{2}\right]$$

$$E(x,t) = e^{ik_c(x - v_{ph}t)} \frac{e^{-(x - v_g t)^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}} = e^{ik_c(x - v_{ph}t)} N(x - v_g t, \sigma)$$

d)
$$v_g = \left(\frac{dk}{d\omega}\right)^{-1} = \frac{c}{n} \left(1 + \frac{d \log n}{d \log \omega}\right)^{-1}$$

For the dilute gas, $n = \sqrt{1 + \alpha} \approx 1 + \frac{\alpha}{2} = n_r + i n_i$ (α is complex)

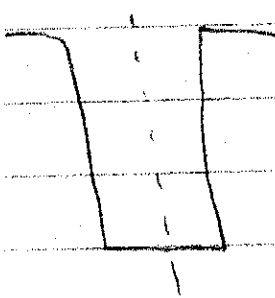
$$\Rightarrow n_r \approx 1 + \frac{n_a e^2}{2\epsilon_0 m \epsilon} \frac{(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2}, \quad n_i \approx \frac{n_a e^2}{2\epsilon_0 m \epsilon} \frac{\gamma \omega}{(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2}$$

$$\Rightarrow n_r = 1 \text{ at } \omega = \omega_0, \quad \frac{d \log n_r}{d \log \omega} = -\frac{n_a e^2}{\epsilon_0 m \epsilon \gamma^2} \text{ at } \omega = \omega_0$$

$$\Rightarrow v_g = 1 + \frac{n_a e^2}{\epsilon_0 m \epsilon \gamma^2} > 1 \text{ at } \omega = \omega_0! \quad \text{This is called anomalous dispersion. It does not violate causality}$$

because signals (information) cannot travel faster than the minimum of (v_{ph}, v_g) and now $v_{ph} = c$ ($n_r = 1$). Also, the waves are damped by the electronic resonance maximally at $\omega = \omega_0$.

Fall 2002 General Exam Part 2 Quantum
Problem 1.



$$-w \leq x \leq w$$

$$x > w$$

$$\psi(x) = \cos kx$$

$$\psi(x) = Ae^{-\alpha x}$$

Match ψ : $\cos kw = Ae^{-\alpha w}$

$$\frac{d\psi}{dx}: -k \sin kw = -\alpha Ae^{-\alpha w}$$

$$k \tan kw = \alpha$$

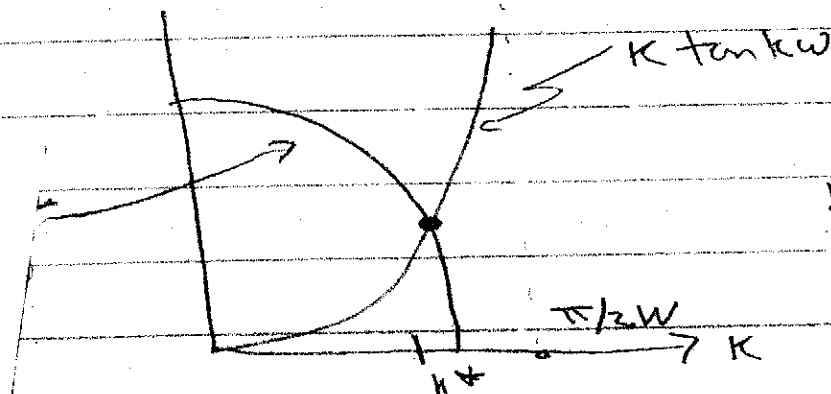
$$E = \frac{\hbar^2 k^2}{2m}$$

$$E = -\frac{\hbar^2 \alpha^2}{2m} + V_0$$

$$\frac{\hbar^2 k^2}{2m} = -\frac{\hbar^2 \alpha^2}{2m} + V_0$$

$$\alpha^2 = \frac{2mV_0}{\hbar^2} - k^2$$

$$k \tan kw = \left(\frac{2mV_0}{\hbar^2} - k^2 \right)^{1/2}$$



$$E = \frac{\hbar^2 k^{*2}}{2m}$$

