Solutions Mechanics 1 SPRING 2001

Kinetic energy

$$T = \frac{m_1}{2}\left(\dot{x}_D^2 + \dot{y}_D^2\right) + \frac{I}{2}\dot{\varphi}_1^2 + \frac{m_2}{2}\left(\dot{x}_c^2 + \dot{y}_c^2\right)$$

Potential energy

$$V = -m_1 g y_D - m_2 g y_c$$

Geometry

$$x_D = a \sin\varphi_1 \qquad y_D = a\cos\varphi_1$$

$$x_c = b\sin\varphi_1 + c\sin\varphi_2 \qquad y_c = b\cos\varphi_1 + c\cos\varphi_2$$

$$\mathcal{L} = T - V$$

$$\mathcal{L} = \frac{\dot{\varphi}_1^2}{2}\left(m_1 a^2 + I + m_2 b^2\right) + \frac{\dot{\varphi}_2^2}{2} \cdot m_2 c^2$$

$$+ \; m_2 b c \,\dot{\varphi}_1 \dot{\varphi}_2 \cos(\varphi_1 - \varphi_2)$$

$$+ \left(m_1 a + m_2 b\right) g \cos\varphi_1 + m_2 c g \cos\varphi_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\varphi_1}\right) = \frac{\partial L}{\partial \varphi_1} \qquad (1)$$

$$(m_1 a^2 + I + m_2 b^2)\ddot\varphi_1 + m_2 bc\,\ddot\varphi_2 \cos(\varphi_1-\varphi_2) + m_2 bc\,\dot\varphi_2^2 \sin(\varphi_1-\varphi_2)$$
$$= -(m_1 a + m_2 b)\, g \sin\varphi_1$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\varphi_2}\right) = \frac{\partial L}{\partial \varphi_2} \qquad (2)$$

$$m_2 c^2 \ddot\varphi_2 + m_2 bc\,\ddot\varphi_1 \cos(\varphi_1-\varphi_2) - m_2 bc\,\dot\varphi_1^2 \sin(\varphi_1-\varphi_2)$$
$$= -m_2 c g \sin\varphi_2$$

Simultaneous motion $\varphi_1 = \varphi_2 = \varphi$ $\quad \cos(\varphi_1-\varphi_2)=1$

$\sin(\varphi_1-\varphi_2)=0$

(1) : $(m_1 a^2 + I + m_2 b^2 + m_2 bc)\,\ddot\varphi = -(m_1 a + m_2 b)\, g \sin\varphi$

(2) : $(m_2 c^2 + m_2 bc)\,\ddot\varphi = -m_2 c g \sin\varphi$

These two equations are satisfied simultaneously if

$$\frac{m_2 c^2 + m_2 bc}{m_1 a^2 + m_2 b^2 + m_2 bc + I} = \frac{m_2 c g}{(m_1 a + m_2 b) g}$$

or

$$\frac{(m_1 a + m_2 b)(b+c)}{m_1 a^2 + m_2 b^2 + m_2 bc + I} = 1$$

Solutions Mechanics 2

$x_1, x_2, x_3 \to$ deviation from equilibrium

$$\mathcal{L} = \frac{m}{2}\left(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2\right) - \frac{k}{2}\left((x_1-x_2)^2 + (x_2-x_3)^2 + (x_3-x_1)^2\right)$$

EQN of motion:

$$m\ddot{x}_1 = -k(x_1-x_2) + k(x_3-x_1)$$
$$m\ddot{x}_2 = +k(x_1-x_2) - k(x_2-x_3)$$
$$m\ddot{x}_3 = +k(x_2-x_3) - k(x_3-x_2)$$

Assume $x_i = A_i \cos(\omega t + \varphi)$

$$\begin{pmatrix} 2k - m\omega^2 & -k & -k \\ -k & 2k-m\omega^2 & -k \\ -k & -k & 2k-m\omega^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = 0$$

$$\mathrm{Det}(M) = 0 \qquad \left(\frac{m\omega^2}{k}\right)\left(\frac{m\omega^2}{k} - 3\right)^2 = 0$$

$$\omega_1 = 0 \qquad \omega_2 = \omega_3 = \sqrt{\frac{3k}{m}}$$

$$\omega_1 = 0 \quad \Rightarrow \quad x_1 = x_2 = x_3 \qquad x_i(t) = at + b$$

Normal Coordinates:

$$q = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad q_1 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} (at + b)$$

q_2, q_3 are ∞ many → orthogonal to q_1 and orthogonal to each other. For example

$$q_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \cos(\omega t + \varphi) \quad q_3 = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \end{pmatrix} \cos(\omega t + \varphi)$$

$$x_1(t) = \left(\frac{1}{\sqrt{3}} q_1 + \frac{1}{\sqrt{2}} q_2 + \frac{1}{\sqrt{6}} q_3\right) A_1$$

$$x_2(t) = \left(\frac{1}{\sqrt{3}} q_1 - \frac{1}{\sqrt{2}} q_2 + \frac{1}{\sqrt{6}} q_3\right) A_2$$

$$x_3(t) = \left(\frac{1}{\sqrt{3}} q_1 - \frac{2}{\sqrt{6}} q_3\right) A_3$$

Initial conditions:

$$x_1(t=0) = \delta \qquad x_2 = x_3 = 0$$

$$\dot{x}_1 = \dot{x}_2 = \dot{x}_3 = 0$$

$$q(t=0) = \begin{pmatrix} \delta \\ 0 \\ 0 \end{pmatrix} \qquad \dot{q}(t=0) \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{matrix} \varphi = 0 \\ a = 0 \\ b = 1 \end{matrix}$$

$$q(t) = \frac{\delta}{\sqrt{3}} \cdot q_1(t) + \frac{\delta}{\sqrt{2}} q_2(t) + \frac{\delta}{\sqrt{6}} \cdot q_3(t)$$

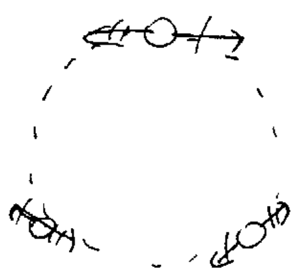
$$q(t=0) = \begin{pmatrix} \delta \\ 0 \\ 0 \end{pmatrix} \qquad \dot{q}(t=0) = 0 \qquad \text{O.K.}$$

$$x_1(t) = \frac{1}{\sqrt{3}} \frac{\delta}{3} + \frac{1}{\sqrt{2}} \frac{\delta}{\sqrt{2}} \cos(\omega t) + \frac{1}{\sqrt{6}} \frac{\delta}{\sqrt{6}} \cos(\omega t)$$

$$= \frac{\delta}{3} + \frac{2\delta}{3} \cos(\omega t)$$

$$x_2(t) = \frac{\delta}{3} - \frac{\delta}{3} \cos(\omega t)$$

$$x_3(t) = \frac{\delta}{3} - \frac{\delta}{3} \cos(\omega t)$$



E & M 1 Solutions

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} \qquad \vec{B} = \mu_0 (\vec{H} + \vec{M})$$

$$\vec{P} = \gamma \vec{\nabla} \times \vec{E} \qquad \vec{M} = 0$$

$$\vec{D} = \varepsilon_0 \vec{E} + \gamma \vec{\nabla} \times \vec{E} \qquad \vec{B} = \mu_0 \vec{H}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \frac{\partial \vec{D}}{\partial t} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \gamma \vec{\nabla} \times \left(\frac{\partial \vec{E}}{\partial t} \right)$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \underbrace{\vec{\nabla} \times (\vec{\nabla} \times \vec{E})}_{= -\nabla^2 \vec{E} + \vec{\nabla}(\underbrace{\vec{\nabla} \cdot \vec{E}}_{=0})} = -\vec{\nabla} \times \left(\frac{\partial \vec{B}}{\partial t} \right)$$

$$-\nabla^2 \vec{E} = -\underbrace{\mu_0 \varepsilon_0}_{\frac{1}{c^2}} \frac{\partial^2 \vec{E}}{\partial t^2} - \gamma \mu_0 \vec{\nabla} \times \left(\frac{\partial^2 \vec{E}}{\partial t^2} \right)$$

Plane wave $$\vec{E} = \vec{E}_0 e^{i(kz - \omega t)} = E_x \hat{x} + E_y \hat{y}$$

$$k^2 \vec{E} = \frac{\omega^2}{c^2} \vec{E} + i \gamma \mu_0 \omega^2 k (\hat{z} \times \vec{E})$$

Component equations:

$$\left(k^2 - \frac{\omega^2}{c^2}\right) E_x + i\,\gamma \mu_0 \omega^2 k \cdot E_y = 0$$

$$-i\,\gamma \mu_0 \omega^2 k \cdot E_x + \left(k^2 - \frac{\omega^2}{c^2}\right) E_y = 0$$

Non zero solutions if:

$$\left(k^2 - \frac{\omega^2}{c^2}\right)^2 - \gamma^2 \mu_0^2 \omega^4 k^2 = 0$$

$$k^2 - \frac{\omega^2}{c^2} = \pm\, \gamma \mu_0 \omega^2 k$$

$$k^2 \mp \gamma \mu_0 \omega^2 \cdot k - \frac{\omega^2}{c^2} = 0$$

$$k = \frac{1}{2}\left[\pm\, \gamma \mu_0 \omega^2 \pm \sqrt{\gamma^2 \mu_0^2 \omega^4 + \frac{4\omega^2}{c^2}}\right]$$

Choose $k > 0$ only

$$k_\pm = \frac{1}{2}\left[\pm\, \gamma \mu_0 \omega^2 + \sqrt{\gamma^2 \mu_0^2 \omega^4 + \frac{4\omega^2}{c^2}}\right]$$

$$n_\pm = \frac{c}{\omega} k_\pm = \pm \frac{\gamma c \mu_0 \omega}{2} + \sqrt{1 + \frac{(\gamma c \mu_0 \omega)^2}{2}}$$

$$n_\pm \simeq 1 \pm \frac{\gamma c \mu_0 \omega}{2}$$

Fields that diagonalize the component matrix are circularly polarized waves

$$E_+ = E_x + i E_y \qquad \text{right-handed}$$

$$E_- = E_x - i E_y \qquad \text{left-handed}$$

E & M Solutions 2

$$\vec{E}\,(r \to \infty) = E_0 \hat{z} = E_0 \left(\hat{r} \cos\theta - \hat{\theta} \sin\theta \right)$$

Potential that will satisfy all boundary conditions is

$$V(r,\theta) = \frac{A}{r} + \left(Br + \frac{C}{r^2} \right) \cos\theta$$

$$\vec{E} = -\vec{\nabla} V = \frac{A}{r^2}\,\hat{r} - \left(B - \frac{2C}{r^3} \right) \cos\theta \cdot \hat{r}$$

$$+ \left(B + \frac{C}{r^3} \right) \sin\theta \cdot \hat{\theta}$$

$r \to \infty \qquad B = -E_0$

$V(r = R) = \text{const} \implies BR + \frac{C}{R^2} = 0$

θ independent

$C = E_0 \cdot R^3$

$$A = \frac{Q}{4\pi\varepsilon_0}$$

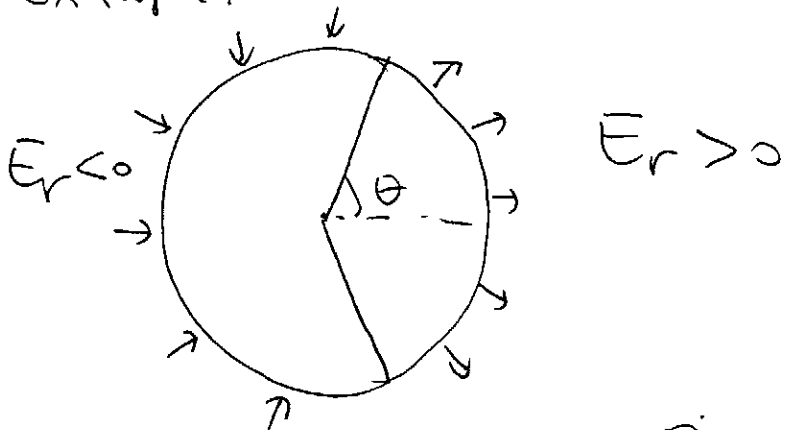
$$\vec{E} = \left[\frac{Q}{4\pi\varepsilon_0 r^2} + E_0 \left(1 + \frac{2R^3}{r^3} \right) \cos\theta \right] \hat{r}$$

$$- \left[E_0 \left(1 - \frac{R^3}{r^3} \right) \sin\theta \right] \hat{\theta}$$

at $r = R$

$$\vec{E} = \underbrace{\left[\frac{Q}{4\pi\varepsilon_0 R^2} + 3E_0\cos\theta\right]}_{E_r}\hat{r}$$

Example:



$E_r < 0$ if $\dfrac{Q}{4\pi\varepsilon_0 R^2} + 3E_0\cos\theta < 0$

$-1 \leq \cos\theta \leq 1$

$Q > 12 E_0 \pi \varepsilon_0 R^2 = Q_{max}$ $\quad E_r < 0$ $\boxed{\text{Never!}}$

$Q < -Q_{max}$ $\quad E_r < 0$ $\boxed{\text{Always}}$

$-Q_{max} < Q < Q_{max}$ $\quad E_r \quad 0$ $\boxed{\text{if } \cos\theta < -\dfrac{Q}{Q_{max}}}$

c) $Q_0 < Q_{max} = 12 E_0 \pi \varepsilon_0 R^2$

Must have $E_r < 0$ for the dust to reach surface

d) $j_r = \rho_0 \mu E_r (r = R)$

$$= 3 \rho_0 \mu E_0 \left(\frac{Q}{Q_{max}} + \cos\theta \right) \qquad (E_r < 0!)$$

e) (1) $Q < -Q_{max}$ $\quad E_r < 0$ everywhere!

$-1 \leq \cos\theta \leq 1$

$$\frac{dQ}{dt} = -\int_0^\pi j_r \, 2\pi R^2 \cdot \sin\theta \, d\theta$$

$$= -6\pi \rho_0 \mu E_0 R^2 \int_0^\pi \left(\frac{Q}{Q_{max}} + \cos\theta \right) \sin\theta \, d\theta$$

$$= -6\pi \rho_0 \mu E_0 R^2 (2 + 0) \cdot \frac{Q}{Q_{max}}$$

$$\frac{1}{Q_{max}} \frac{dQ}{dt} = -\frac{\rho_0 \mu}{\varepsilon_0} \frac{Q}{Q_{max}}$$

② $-Q_{max} < Q < Q_{max}$

$$\frac{dQ}{dt} = -6\pi \rho_0 \mu E_0 R^2 \cdot \int_{\theta_c}^{\pi} \left(\frac{Q}{Q_{max}} + \cos\theta \right) \sin\theta \, d\theta$$

$$\theta_c = \arccos\left(\frac{-Q}{Q_{max}} \right) \qquad \cos\theta_c = -\frac{Q}{Q_{max}}$$

$$\frac{1}{Q_{max}} \frac{dQ}{dt} = -\frac{\rho_0 \mu}{2\varepsilon_0} \cdot \left[\frac{Q}{Q_{max}} \left(1 + \cos\theta_c \right) + \frac{1}{2} \left(\cos^2\theta_c - 1 \right) \right]$$

$$= -\frac{\rho_0 \mu}{2\varepsilon_0} \left[\frac{Q}{Q_{max}} \left(1 - \frac{Q}{Q_{max}} \right) + \frac{1}{2} \left(\frac{Q^2}{Q_{max}^2} - 1 \right) \right]$$

$$\frac{1}{Q_{max}} \frac{dQ}{dt} = \frac{\rho_0 \mu}{4\varepsilon_0} \left(1 - \frac{Q}{Q_{max}} \right)^2$$

Solution QM1

a) $$\hat{S}_{\vec{n}} = \frac{\hbar}{2}\begin{bmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{bmatrix}$$

eigenvalues $\pm \hbar/2$

$\hbar/2$ has $\begin{bmatrix} \cos\theta/2\, e^{-i\phi/2} \\ \sin\theta/2\, e^{i\phi/2} \end{bmatrix}$ overall phase not determined

b) let $|\psi\rangle$ be

$$\langle\psi|\hat{S}_z|\psi\rangle = \frac{\hbar}{2}\left[\cos^2\theta/2 - \sin^2\theta/2\right] = 0$$

$\Rightarrow \cos\theta/2 = \pm\sin\theta/2$ say + (will also work for −)

Now $$|\psi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} e^{-i\phi/2} \\ \pm e^{i\phi/2} \end{bmatrix}$$

$\langle\psi|\hat{S}_x|\psi\rangle$ proportional to $e^{-i\phi} \pm e^{i\phi}$

$\langle\psi|\hat{S}_y|\psi\rangle$ proportional to $e^{-i\phi} - e^{i\phi}$

Both can not be zero!

Solution QM1 c)

$$U(t) = e^{-i\hat{H}t/\hbar}$$

$$\hat{H} = -g\mu B \frac{\hbar}{2}\sigma_y \qquad \frac{\hat{H}}{\hbar} = -g\frac{\mu B}{2}\sigma_y$$

$$\gamma \equiv g\frac{\mu B}{2} \qquad \frac{\hat{H}}{\hbar} = -\gamma\sigma_y$$

$$U(t) = e^{i\gamma t\sigma_y}$$

$$= \cos\gamma t + i\sigma_y \sin\gamma t$$

$$= \begin{bmatrix} \cos\gamma t & \sin\gamma t \\ -\sin\gamma t & \cos\gamma t \end{bmatrix}$$

$$|\psi(0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |\psi(T)\rangle = \begin{bmatrix} \cos\gamma T \\ -\sin\gamma T \end{bmatrix}$$

$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ eigenvector with $\hbar/2$ $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

probability $\boxed{\frac{1}{2}\left(\cos(\gamma T) - \sin(\gamma T)\right)^2}$

QM2 Solution

$\hat{T}(d) = e^{i\hat{p}d/\hbar}$ ← translates by d

Want $\langle j | T(d) | 0 \rangle$

$$\hat{a} - \hat{a}^{+} = \frac{1}{\sqrt{2\hbar\omega}} (i\hat{p} + i\hat{p}) = \frac{\sqrt{2}}{\sqrt{\hbar\omega}} i \hat{p}$$

$$\hat{p} = \frac{\sqrt{\hbar\omega}}{\sqrt{2}\, i} (\hat{a} - \hat{a}^{+}) \qquad \frac{i\hat{p}d}{\hbar} = \frac{\sqrt{w}\, d}{\sqrt{\hbar}\sqrt{2}} (\hat{a} - \hat{a}^{+}) = r(\hat{a} - \hat{a}^{+})$$

$$r = \frac{\sqrt{\omega}\, d}{\sqrt{\hbar}\sqrt{2}}$$

$$e^{r(\hat{a} - \hat{a}^{+})} = ? \qquad \hat{A} = -r\hat{a}^{+} \quad \hat{B} = r\hat{a}$$

$$[\hat{A}, \hat{B}] = -r^2 [\hat{a}^{+}, \hat{a}] = r^2$$

$$e^{(r\hat{a} - r\hat{a}^{+})} = e^{-r\hat{a}^{+}} e^{r\hat{a}} e^{-\frac{1}{2}r^2}$$

$$\langle j | T(d) | 0 \rangle = e^{-\frac{1}{2}r^2} \langle j | e^{-r\hat{a}^{+}} e^{r\hat{a}} | 0 \rangle$$

$$= e^{-\frac{1}{2}r^2} \langle j | e^{-r\hat{a}^{+}} | 0 \rangle = e^{-\frac{1}{2}r^2} \sum_{\ell=0}^{\infty} \langle j | \frac{(-r\hat{a}^{+})^{\ell}}{\ell!} | 0 \rangle$$

$$= e^{-\frac{1}{2}r^2} \frac{(-r)^j}{\sqrt{j!}} \qquad \boxed{P(j) = e^{-r^2} \frac{r^{2j}}{j!}}$$

b) $|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi(0)\rangle$

$$= \sum_n e^{-i\omega(n+1/2)t} |n\rangle\langle n|\psi(0)\rangle$$

Probabilities will be the same for

$$t = \frac{2\pi}{\omega}, \frac{4\pi}{\omega}, \ldots$$

c) Probability of finding any ~~energy~~ energy is independent of t.

Stat Mech 1 Solution

$$dU = dQ - dW$$

Heat absorbed — work done by gas

Step 1. Isothermal $dU = 0$

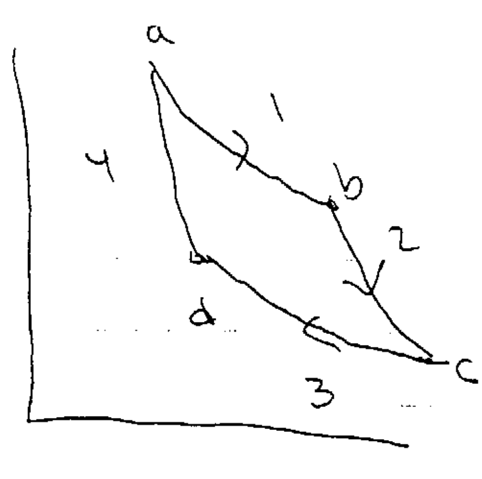
$$dQ = dW$$

$$dQ = p\,dV = \frac{NkT}{V}\,dV$$

$$|Q_H| = NkT_H \ln\left(V_b/V_a\right)$$

tep 3. $|Q_L| = NkT_L \ln\left(V_c/V_d\right)$

along 2 and 4 $dQ = 0$



$$dU = -p\,dV$$

$$\frac{f}{2} Nk\,dT = -p\,dV$$

$$\frac{f}{2}(p\,dV + V\,dp) = -p\,dV$$

$$\frac{f}{2} V\,dp = \left(1 + \frac{f}{2}\right) p\,dV$$

$$\frac{dp}{p} = \alpha \frac{dV}{V}$$

Stat Mech I solution.

$$\frac{P_c}{P_b} = \left(\frac{V_c}{V_b}\right)^{\alpha} \qquad \frac{P_a}{P_d} = \left(\frac{V_a}{V_d}\right)^{\alpha}$$

a and b have the same energy

$$P_a V_a = P_b V_b \qquad \text{also} \qquad P_c V_c = P_d V_d$$

$$\frac{P_a}{P_b} = \frac{V_b}{V_a} \qquad\qquad \frac{P_c}{P_d} = \frac{V_d}{V_c}$$

$$\frac{P_a P_c}{P_b P_d} = \frac{V_b V_d}{V_a V_c}$$

$$\frac{P_a P_c}{P_b P_d} = \left(\frac{V_c V_a}{V_b V_d}\right)^{\alpha}$$

$$\frac{V_b V_d}{V_a V_c} = \frac{V_c^{\alpha} V_a^{\alpha}}{V_b^{\alpha} V_d^{\alpha}} \qquad \frac{V_b^{\alpha+1}}{V_a^{\alpha+1}} = \frac{V_c^{\alpha+1}}{V_d^{\alpha+1}}$$

$$\Rightarrow \frac{V_b}{V_a} = \frac{V_c}{V_d}$$

$$e = 1 - \frac{|Q_L|}{|Q_H|} = 1 - \frac{T_L}{T_H}$$

Stat Mech 2 Solutions

a) $$\frac{N}{2} = V\int_{k<k_f} \frac{d^3k}{(2\pi)^3} = V\int_0^{k_f} \frac{4\pi}{(2\pi)^3} k^2 dk = \frac{V k_f^3}{6\pi^2}$$

$$k_f = (3\pi^2 n)^{1/3}$$

$$E_0 = 2V\int_{k<k_f} \frac{\hbar^2 k^2}{2m} \frac{d^3k}{(2\pi)^3} = 2V \frac{\hbar^2}{2m} \frac{4\pi}{(2\pi)^3} \frac{k_f^5}{5} = \frac{V}{5\pi^2} \frac{\hbar^2}{2m} k_f^5$$

$$\frac{E_0}{V} = \frac{3}{5} (3\pi^2)^{2/3} \frac{\hbar^2}{2m} n^{5/3}$$

b) $$k_{f\pm} = (6\pi^2 n_\pm)^{1/3}$$

c) $$\frac{E_{kin}}{V} = \frac{1}{10\pi^2} \frac{\hbar^2}{2m} (k_{f+}^5 + k_{f-}^5)$$

$$= \frac{1}{10\pi^2} \frac{\hbar^2}{2m} (6\pi^2)^{5/3} (n_+^{5/3} + n_-^{5/3})$$

$$= \frac{3}{5} (6\pi^2)^{2/3} \frac{\hbar^2}{2m} (n_+^{5/3} + n_-^{5/3})$$

$$= \frac{E_0}{V} + \frac{4}{3} (3\pi^2)^{2/3} \frac{\hbar^2 n^{-1/3}}{2m} \delta^2 + \ldots$$

d) $\frac{U}{V} = \alpha n_+ n_- = \alpha\left(\frac{n}{2}+\delta\right)\left(\frac{n}{2}-\delta\right)$

$= \alpha \frac{n^2}{4} - \alpha\delta^2$

e)

$$\frac{E}{V} = \frac{E_0}{V} + \frac{\alpha n^2}{4} + \left[\frac{4}{3}(3\pi^2)^{2/3}\frac{\hbar^2 n^{-1/3}}{2m} - \alpha\right]\delta^2 + \ldots$$

$$\alpha_c = \frac{4}{3}(3\pi^2)^{2/3}\frac{\hbar^2 n^{-1/3}}{2m}$$