MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF PHYSICS

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DOCTORAL GENERAL EXAMINATION

WRITTEN EXAM — WITH SOLUTIONS

September 4, 2015

DURATION: 75 MINUTES PER SECTION

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DOCTORAL GENERAL EXAMINATION

WRITTEN EXAM - CLASSICAL MECHANICS — WITH SOLUTIONS

September 4, 2015

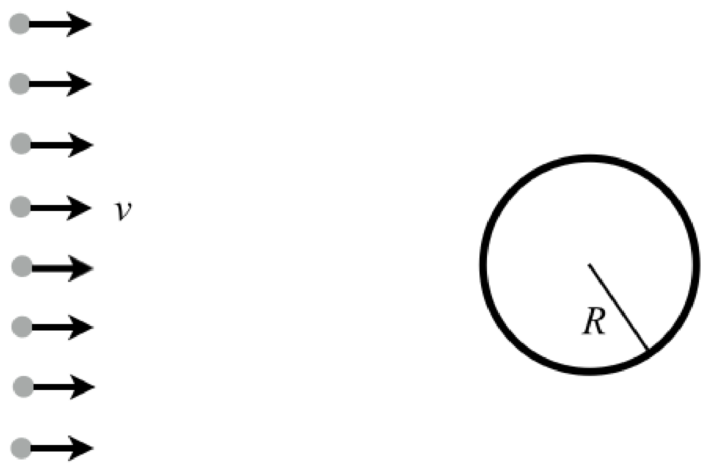
DURATION: 75 MINUTES

1. This examination has two problems. Read both problems carefully before making your choice. Submit ONLY one problem. IF YOU SUBMIT MORE THAN ONE PROBLEM FROM A SECTION, BOTH WILL BE GRADED, AND THE PROBLEM WITH THE LOWER SCORE WILL BE COUNTED.
2. If you decide at this time not to take this section of the exam, please inform the faculty proctor. ONCE YOU BEGIN THE EXAM, IT WILL BE COUNTED.
3. Calculators may not be used.
4. No books or reference materials may be used.

Classical Mechanics 1: Striking the sphere

A hard sphere of radius R is fixed at the center of a spherically symmetric potential $U(r)$. The potential $U(r)$ declines monotonically to zero as $r \longrightarrow \infty$.

A beam of test particles is aimed at the sphere from a great distance. All the particles are moving parallel to one another with speed $v \ll c$. The number density of particles is n particles/cm^3, and each particle has a mass m.



(a) (*4 pts*) Calculate the cross-section for striking the sphere, in terms of $U(R)$, v, and m.

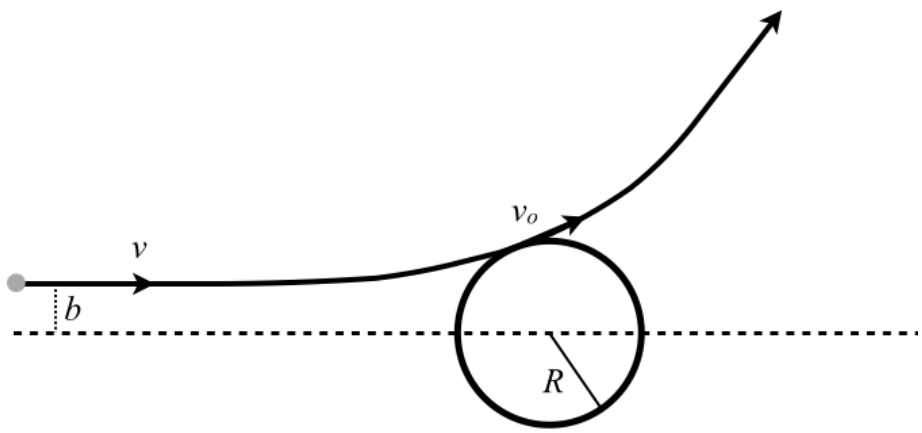
(b) (*2 pts*) Is it possible for the cross-section to vanish? If so, under what conditions?

(c) (*4 pts*) Now take $U(r) = -GMm/r$, the gravitational potential between the sphere of mass M and the test particle of mass m.

Whenever a particle strikes the sphere, the particle sticks to the sphere and increases the sphere's mass but without appreciably changing the radius. Calculate the time required for the sphere's mass to increase from an initial value of M_i to a final value of M_f.

- Solutions by J. Winn (August 2015). An earlier version of this problem also appeared on the Fall 1992 exam.

  (a) The figure below illustrates the case of a repulsive potential, although the result is the same for an attractive potential. Consider the particles with impact parameter b just large enough for the point of closest approach to equal R. All particles with smaller impact parameters will collide with the sphere.

  To figure out b in terms of given quantities, use the conservation of energy,

  $$\frac{1}{2}mv^2 = \frac{1}{2}mv_0^2 + U(R), \tag{1}$$

  as well as the conservation of angular momentum,

  $$mvb = mv_0R, \tag{2}$$

  giving $v_0 = v(b/R)$. Combining these results,

  $$\begin{aligned} \frac{1}{2}mv^2 &= \frac{1}{2}mv^2\left(\frac{b}{R}\right)^2 + U(R) & (3)\\ 1 &= \left(\frac{b}{R}\right)^2 + \frac{2U(R)}{mv^2} & (4)\\ b^2 &= R^2\left[1 - \frac{2U(R)}{mv^2}\right]. & (5) \end{aligned}$$

  The cross-section for hitting the sphere is therefore

  $$\sigma = \pi b^2 = \pi R^2\left[1 - \frac{2U(R)}{mv^2}\right]. \tag{6}$$

(b) From our previous answer we see that for the cross-section to vanish, $U(R)$ must be positive. Specifically, σ vanishes when

$$\frac{1}{2}mv^2 = U(R), \tag{7}$$

and the particles with $b = 0$ have just enough energy to touch the sphere.

(c) In time dt, the number of particles hitting the sphere is $nv\sigma dt$. Therefore

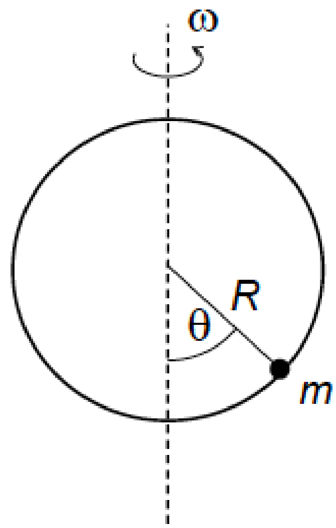
$$\begin{aligned} \frac{dM}{dt} &= mnv\sigma & (8) \\ \frac{dM}{dt} &= mnv\pi R^2 \left[1 + \frac{2GMm}{mv^2R}\right] & (9) \\ \frac{dM}{dt} &= K\left[1 + \frac{M}{M_0}\right], & (10) \end{aligned}$$

where we have defined $K \equiv mnv\pi R^2$ and $M_0 \equiv v^2R/2G$. Now we may integrate:

$$\begin{aligned} \frac{dM}{1 + M/M_0} &= Kdt & (11) \\ \int_{M_i}^{M_f} \frac{dM}{1 + M/M_0} &= Kt & (12) \\ M_0 \ln(M_0 + M_f) - M_0 \ln(M_0 + M_i) &= Kt & (13) \\ t &= \frac{M_0}{K} \ln\left(\frac{M_0 + M_f}{M_0 + M_i}\right) & (14) \\ t &= \frac{v}{2\pi GmnR} \ln\left(\frac{v^2R/2G + M_f}{v^2R/2G + M_i}\right). & (15) \end{aligned}$$

Classical Mechanics 2: Particle sliding on a rotating circular wire

A pointlike particle of mass m is constrained to move on a circular wire of radius R. The particle can slide without friction. The circular wire spins with constant angular speed ω about the vertical diameter. The force of gravity mg is acting on the particle.



(a) (*2 pts*) Write the Lagrangian for the system using θ as the generalized coordinate. Identify an effective potential $V(\theta)$.

(b) (*2 pts*) Write down the Euler Lagrange equation and derive the equation of motion in terms of m, g, R, ω, and θ. (Do not solve it.)

(c) (*2 pts*) Find constant values $\theta_i, i = 1, 2, ...,$ for which $\theta(t) = \theta_i$ is a stationary solution of the equation of motion. Express your answer in terms of ω, R, and g. Do all solutions exist for all values of ω?

(d) (*4 pts*) Consider now small displacements from each of the equilibrium values θ_i identified in part (c). Determine, as a function of ω whether such displacements lead to stable or unstable oscillations. If the small-amplitude oscillation is stable, determine the corresponding oscillation frequency Ω_i. Summarize your results in a graph where you plot the θ_i as functions of ω and label the various parts of the curves as "stable" or "unstable".

- Solutions by Markus Klute (August 2015) An earlier version of this problem also appeared on the Spring 2009 exam.

  (a) The velocity $\vec{v}$ of the particle is

$$\vec{v} = R\,\dot{\theta}\,\vec{e}_\theta + \omega R \sin\theta\,\vec{e}_\phi\,. \tag{1}$$

  The Lagrangian L is then given by

$$L = T - V = \frac{1}{2}m\left(R\,\dot{\theta}\,\vec{e}_\theta + \omega R\sin\theta\,\vec{e}_\phi\right)^2 - mgR\,(1-\cos\theta)\;. \tag{2}$$

  Simplifying and dropping constants

$$L = \frac{1}{2}m\,R^2\,\dot{\theta}^2 + \frac{1}{2}m\,\omega^2 R^2\sin^2\theta + mgR\cos\theta = \frac{1}{2}m\,R^2\,\dot{\theta}^2 - V(\theta)\,, \tag{3}$$

  with

$$V(\theta) = -\frac{1}{2}m\,\omega^2 R^2\sin^2\theta - mgR\cos\theta\,. \tag{4}$$

  (b) The Euler Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = \frac{\partial L}{\partial\theta} \tag{5}$$

  gives

$$\frac{1}{2}m\,R^2 2\ddot{\theta} = -mgR\sin\theta + m\omega^2R^2\sin\theta\cos\theta\,. \tag{6}$$

  Canceling out constants results in

$$\ddot{\theta} = -\frac{g}{R}\sin\theta + \omega^2\sin\theta\cos\theta\,. \tag{7}$$

  (c) If θ is constant then $\ddot{\theta} = 0$, so from the part (b) of this problem we get

$$\sin\theta\left(\omega^2\cos\theta - \frac{g}{R}\right) = 0\,. \tag{8}$$

  There are three solutions

$$\theta_1 = 0\,, \tag{9}$$
$$\theta_2 = \cos^{-1}\left(\frac{g}{R\omega^2}\right)\,, \tag{10}$$
$$\theta_3 = \pi\,. \tag{11}$$

  The solutions θ_1 and θ_3 exist for all values of ω. The solution θ_2 exists for

$$\omega^2 > \frac{g}{R} \equiv \omega_0^2\,. \tag{12}$$

(d) From the equation of motion we find

$$m\,R^2\ddot{\theta} = \frac{\partial L}{\partial \theta} = -\frac{\partial V}{\partial \theta}. \tag{13}$$

Around the equilibrium points θ_i we write $\theta = \theta_i + \epsilon$ and from $\frac{\partial V}{\partial \theta}|_{\theta_i}$ we find

$$m\,R^2\,\ddot{\epsilon} = -\left.\frac{\partial^2 V}{\partial \theta^2}\right|_{\theta_i} \epsilon. \tag{14}$$

The oscillation frequency Ω_i is therefore

$$\Omega_i^2 = \frac{1}{mR^2}\left.\frac{\partial^2 V}{\partial \theta^2}\right|_{\theta_i}. \tag{15}$$

Using the potential from part (a) gives

$$\frac{\partial^2 V}{\partial \theta^2} = -m\,\omega^2 R^2(\cos^2\theta - \sin^2\theta) + m\,g\,R\cos\theta\,. \tag{16}$$

We therefore find

$$\Omega_i^2 = -\omega^2(2\cos^2\theta_i - 1) + \omega_0^2\cos\theta_i\,. \tag{17}$$

For $\theta_1 = 0$ we find

$$\Omega_i^2 = \omega_0^2 - \omega^2\,. \tag{18}$$

This point is stable for $\omega < \omega_0$ and unstable for $\omega > \omega_0$.

For θ_2 satisfying $\cos\theta_2 = \frac{\omega_0^2}{\omega^2}$, we find

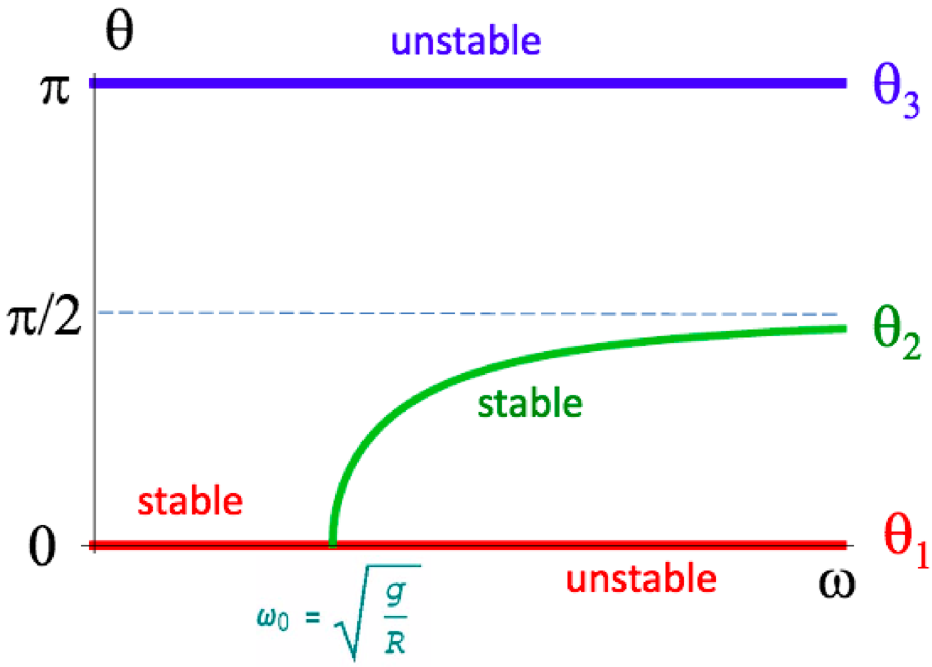
$$\Omega_i^2 = -\omega^2\left(\frac{2\omega_0^4}{\omega^4} - 1\right) + \frac{\omega_0^4}{\omega^2} = \omega^2 - \frac{\omega_0^4}{\omega^2}\,. \tag{19}$$

This is stable for $\omega > \omega_0$.

For $\theta_3 = \pi$ we find

$$\Omega_i^2 = -\omega_0^2 - \omega^2 < 0\,, \tag{20}$$

which is always unstable.


θ
unstable
π
θ_3
π/2
θ_2
stable
stable
0
θ_1
unstable
ω
$\omega_0 = \sqrt{\frac{g}{R}}$

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DOCTORAL GENERAL EXAMINATION

WRITTEN EXAM - ELECTRICITY AND MAGNETISM — WITH SOLUTIONS

September 4, 2015

DURATION: 75 MINUTES

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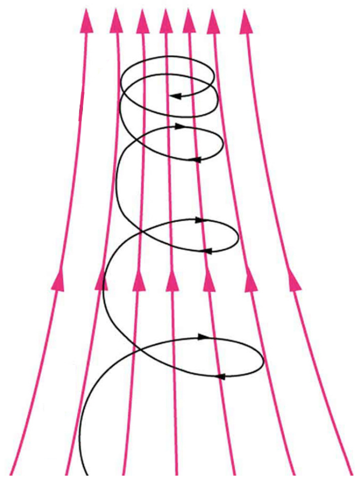
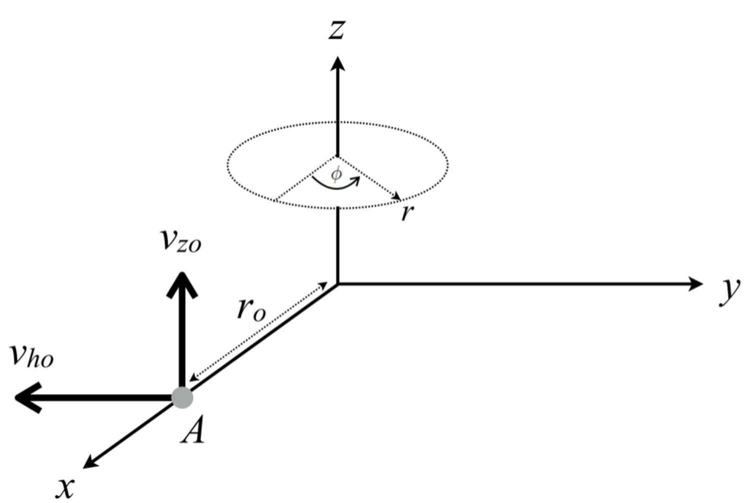
Electromagnetism 1: Magnetic mirror

When a charged particle orbits around magnetic field lines while also drifting along the field into a region of higher field strength, then the particle experiences a force that reduces the component of velocity parallel to the field. This force slows the motion along the field lines and may reverse it. This is the basis of a "magnetic mirror," illustrated below.



In this problem we will investigate this phenomenon using a cylindrical coordinate system in which the z-axis is the symmetry axis, r denotes the cylindrical radius (the distance from the z-axis), and ϕ is the azimuthal angle measured from the x-axis, as illustrated below.



You may find it useful to remember

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}. \tag{1}$$

Consider a magnetic field $\vec{B}$ which is axially symmetric around the z-axis. The z-component is

$$B_z(r,z) = B_0 + B_0' z, \tag{2}$$

where B_0' is a constant.

We inject a particle with mass m and charge $q > 0$ at point A, located at $x = r_0$ and $y = z = 0$. The particle's initial velocity has both horizontal and vertical components:

$$\vec{v} = -v_{\text{h}0}\hat{y} + v_{\text{z}0}\hat{z}, \tag{3}$$

where $v_{\text{h}0} = \frac{q}{mc} B_0 r_0$ and $v_{\text{z}0} \ll v_{\text{h}0}$. Both components of the velocity are non-relativistic.

(a) (*1 pt*) Calculate the radial component of the magnetic field, $B_r(r,z)$.

(b) (*1 pt*) Show that throughout the subsequent motion,

$$v^2(t) = v_{h0}^2 + v_{z0}^2. \tag{4}$$

Now assume that during each orbit around the z-axis, the horizontal velocity is nearly in the $-\hat{\phi}$ direction and the change in radius Δr of the orbit is negligible compared to the instantaneous radius r.

(c) (*1 pt*) What is the radius r of the orbit as a function of v_ϕ, B_z and physical constants?

(d) (*2 pts*) Find an equation for the z-dependence of v_h (the horizontal speed) as the particle drifts in the z-direction. One way to do so is to write the equation of motion for the $\hat{\phi}$ component of the velocity, and then use $v_z dt = dz$.

(e) (*3 pts*) Using your result from part (c), integrate your equation from part (d) to show that the horizontal speed varies with z as

$$\frac{v_h(z)}{v_{h0}} = \sqrt{\frac{B_z(z)}{B_0}}. \tag{5}$$

(f) (*2 pts*) Find the value of z for which $v_z = 0$. This is the reflection point of the magnetic mirror, where the spiraling particle stops its upward motion and starts moving downward.

Express your answer entirely in terms of B_0, B_0', v_{h0} and v_{z0}.

- Solutions by J. Winn (August 2015), based on a version of this problem that appeared in the Spring 1986 exam.

  (a) Requiring the divergence of $\vec{B}$ to vanish,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rB_r\right) + \frac{\partial B_z}{\partial z} = 0 \quad (6)$$

$$\frac{\partial}{\partial r}\left(rB_r\right) = -B_0' r \quad (7)$$

$$rB_r = -\frac{r^2}{2}B_0' + \text{constant}. \quad (8)$$

  Note that the constant of integration must be zero, for B_r to remain finite at $r = 0$. Therefore

$$B_r = -\frac{rB_0'}{2}. \quad (9)$$

  (b) The only force on the particle is the Lorentz force, which cannot change the kinetic energy of the particle:

$$\vec{F} = \frac{q}{c}\vec{v} \times \vec{B} \quad (10)$$

$$\frac{dE_K}{dt} = \vec{F} \cdot \vec{v} = 0 \quad (11)$$

$$E_K = \frac{1}{2}mv^2 = \frac{1}{2}m(v_h^2 + v_z^2) = \text{constant} \quad (12)$$

  from which the desired result follows.

  (c) Since the motion is nearly circular, we have

$$\frac{mv_h^2}{r} = \frac{qv_h B_z}{c}, \quad (13)$$

  which gives

$$r = \frac{mv_h c}{qB_z}. \quad (14)$$

  (d) The ϕ component of the Lorentz equation of motion is

$$F_\phi = m\frac{dv_\phi}{dt} = \frac{q}{c}\left(v_z B_r - v_r B_z\right), \quad (15)$$

  and since we are assuming a nearly circular orbit, $v_r \approx 0$. Thus

$$\frac{dv_\phi}{dt} = \frac{q}{mc}v_z B_r = -\frac{q}{mc}v_z\frac{rB_0'}{2}. \quad (16)$$

  Dividing both sides by v_z, and using $v_h = -v_\phi$ (the particle circulates in the $-\phi$ direction) we obtain

$$\frac{dv_h}{v_z dt} = \frac{dv_h}{dz} = \frac{q}{mc}\frac{rB_0'}{2}. \quad (17)$$

(e) Replace r by $mv_\phi c/qB_z$ in the preceding equation, giving

$$\frac{dv_h}{dz} = \frac{q}{mc}\frac{mv_h cB_0'}{2qB_z} \tag{18}$$

$$\frac{1}{v_h}\frac{dv_h}{dz} = \frac{1}{2}\frac{B_0'}{B_z}. \tag{19}$$

We integrate to find $v_h(z)$:

$$\int_{v_{h0}}^{v_h} \frac{dv_h}{v_h} = \frac{1}{2}B_0' \int_0^z \frac{dz}{B_0 + B_0' z} \tag{20}$$

$$\ln\left(\frac{v_h}{v_{h0}}\right) = \frac{1}{2}\ln\left[\frac{B_z(z)}{B_0}\right] \tag{21}$$

$$\frac{v_h(z)}{v_{h0}} = \sqrt{\frac{B_z(z)}{B_0}}. \tag{22}$$

(f) Denote by z_r the height of the reflection point, at which $v_z(z_r) = 0$. From part (b),

$$[v_h(z_r)]^2 = v_{h0}^2 + v_{z0}^2. \tag{23}$$

Now we make use of our result from part (e), by writing

$$\left[\frac{v_h(z_r)}{v_{h0}}\right]^2 = 1 + \left(\frac{v_{z0}}{v_{h0}}\right)^2 \tag{24}$$

$$\frac{B_0 + B_0' z_r}{B_0} = 1 + \left(\frac{v_{z0}}{v_{h0}}\right)^2 \tag{25}$$

allowing us to solve for z_r:

$$z_r = \left(\frac{B_0}{B_0'}\right)\left(\frac{v_{z0}}{v_{h0}}\right)^2. \tag{26}$$

Electromagnetism 2: Electromagnetic waves in a plasma

A plane electromagnetic wave of angular frequency ω propagates in a uniform plasma with electron density N_e. The plasma is locally neutral, $\rho = 0$. The electromagnetic wave generates periodic currents within the plasma that, as the problem will show, modify the index of refraction of the medium compared to that of the vacuum.

Assume the plasma has no resistivity and neglect radiation pressure effects as well as currents due to the ions.

(a) (*3 pts*) Relate the current $\vec{J}(\vec{r}, t)$ in the plasma to the wave's electric field $\vec{E}(\vec{r}, t)$ or derivatives thereof. Assume magnetic forces can be neglected.

(b) (*3 pts*) Write down the appropriate Maxwell equations and derive the wave equation. Find the dispersion relation $\omega(k)$ and the lowest frequency electromagnetic wave that can propagate the plasma.

(c) (*2 pts*) Find the phase and group velocity for electromagnetic waves in the plasma. Compare those velocities with c, the speed of light in vacuum.

(d) (*1 pt*) Find the index of refraction n of the plasma as a function of frequency.

(e) (*1 pt*) If a plane electromagnetic wave is incident on a plane interface between a vacuum and the plasma, what is the critical angle for total reflection, measured from the normal to the interface?

- Solutions by Markus Klute (August 2015) An earlier version of this problem also appeared on the Fall 2010 exam.

  (a) As stated in the problem, we ignore the ions and focus on the electrons. The motion of an electron at location $\vec{r}$ is
  $$m\,\ddot{\vec{s}} = -e\vec{E}(\vec{r},t)\,, \tag{1}$$
  where $\vec{s}$ is the small displacement vector of the electron, lying along the direction of the electric field. Because of the harmonic nature of the wave, the solution for $\vec{s}$ will also be harmonic with angular frequency ω so that $\ddot{\vec{s}} = -\omega^2\vec{s}$. We thus get
  $$\vec{s}(\vec{r},t) = \frac{e}{m\omega^2}\vec{E}(\vec{r},t)\,. \tag{2}$$
  It follows that the velocity of the electron is given by
  $$\vec{v}(\vec{r},t) = \frac{e}{m\omega^2}\frac{\partial\vec{E}(\vec{r},t)}{\partial t}\,. \tag{3}$$
  The overall current density in the plasma at $\vec{r}$ is given by
  $$\vec{J} = -eN_e v(\vec{r},t) = -\frac{e^2N_e}{m\omega^2}\frac{\partial\vec{E}(\vec{r},t)}{\partial t}\,. \tag{4}$$

  (b) In Gaussian units the appropriate Maxwell equations are
  $$\begin{aligned}
  \vec{\nabla}\cdot\vec{B} &= 0 && (5)\\
  \vec{\nabla}\cdot\vec{E} &= 4\pi\rho = 0 && (6)\\
  \vec{\nabla}\times\vec{B} &= \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t} && (7)\\
  \vec{\nabla}\times\vec{E} &= -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} && (8)
  \end{aligned}$$
  Note that ρ is taken to be zero due to the condition of local charge neutrality. We now use the results for $\vec{J}$ to simplify Maxwell's equations for the plasma.
  $$\begin{aligned}
  \vec{\nabla}\times\vec{E} &= -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} && (9)\\
  \vec{\nabla}\times\vec{B} &= -\frac{4\pi e^2N_e}{m\,c\,\omega^2}\frac{\partial\vec{E}}{\partial t} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t} && (10)\\
  &= \frac{1}{c}\left(1 - \frac{4\pi e^2N_e}{m\,\omega^2}\right)\frac{\partial\vec{E}}{\partial t} && (11)
  \end{aligned}$$
  We can combine these two equations, while making use of $\vec{\nabla}\cdot\vec{E} = 0$, to find the wave equation in the plasma
  $$\nabla^2\vec{E} = \left(1 - \frac{4\pi e^2N_e}{m\,\omega^2}\right)\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}\,. \tag{12}$$

We can write this as

$$\nabla^2 \vec{E} = \left(1 - \frac{\omega_p^2}{\omega^2}\right) \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}\,, \tag{13}$$

where ω_p is the plasma frequency

$$\omega_p^2 \equiv \left[\frac{4\pi e^2 N_e}{m}\right]_{cgs} \quad \text{or} \quad \left[\frac{e^2 N_e}{\epsilon_0\, m}\right]_{\text{m}ks}\,. \tag{14}$$

The wave equation admits to plane wave solutions with dispersion relation

$$k^2 = \left(1 - \frac{\omega_p^2}{\omega^2}\right) \frac{\omega^2}{c^2}\,, \tag{15}$$

or equivalently,

$$\omega = \sqrt{\omega_p^2 + k^2 c^2}\,. \tag{16}$$

Since k has no real solution for $\omega < \omega_p$, we conclude that the propagating waves must have a frequency above the plasma frequency.

(c) The phase and group velocities are given by

$$v_p = \frac{\omega(k)}{k} \quad \text{and} \quad v_g = \frac{d\omega}{dk}. \tag{17}$$

We thus find

$$v_p = \frac{c}{\sqrt{1 - \frac{\omega_p^2}{\omega^2}}} \quad \text{and} \quad v_g = \frac{kc^2}{\omega} = c\sqrt{1 - \frac{\omega_p^2}{\omega^2}}\,. \tag{18}$$

The group velocity is less than c and the phase velocity larger than c.

(d) The index of refraction n is defined from the phase velocity as $v_p = \frac{c}{n}$. We thus find

$$n = \sqrt{1 - \frac{\omega_p^2}{\omega^2}}\,. \tag{19}$$

Note that $n < 1$.

(e) A plane wave incident on a planar boundary with a medium having an effective index of refraction n is subject to Snell's law

$$n_{\text{vac}} \sin\theta_{\text{inc}} = n_{plasma} \sin\theta_{refr}\,, \tag{20}$$

where the angles are measured with respect to the direction normal to the interface. For our problem, this results in

$$\sin\theta_{\text{inc}} = n \sin\theta_{refr}\,. \tag{21}$$

Since the index of refraction n of the medium is less than unity the refracted angle reaches 90° before the incident angle reaches 90°. Thus, the critical incident angle, after which the incident radiation is totally reflected, is

$$\sin\theta_{critical} = n\,. \tag{22}$$

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DOCTORAL GENERAL EXAMINATION

WRITTEN EXAM - STATISTICAL MECHANICS — WITH SOLUTIONS

September 4, 2015

DURATION: 75 MINUTES

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Statistical Mechanics 1: Potts Model

The q-state Potts model generalizes the Ising model. There is a variable $\sigma_i \in \{1, 2, \ldots, q\}$ at each lattice site. The Hamiltonian is given by a sum over nearest neighbors:

$$H_{\text{Potts}} = -\frac{3J}{2} \sum_{\langle i,j \rangle} \delta_{\sigma_i, \sigma_j}. \tag{1}$$

There are N lattice sites. Assume $J > 0$. For parts (a) and (b), $q \geq 2$ is arbitrary, and for (c) and (d), we assume $q = 3$.

(a) (*1 pt*) What is the entropy of the system at $T = 0$?

(b) (*3 pts*) For this part only, suppose the N sites are arranged on a line with open boundary conditions. Write down the (exact) free energy.

(c) (*1 pt*) For the $q = 3$ case, show that the model is equivalent to

$$H = -J \sum_{\langle i,j \rangle} \vec{s}_i \cdot \vec{s}_j \tag{2}$$

with each $\vec{s}_i$ restricted to take values in the set

$$\vec{s}_i \in \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix}, \begin{pmatrix} -1/2 \\ -\sqrt{3}/2 \end{pmatrix} \right\} \tag{3}$$

(d) (*5 pts*) Let $\vec{m}$ denote the mean-field magnetization $\sum_i \vec{s}_i / N$. Suppose that each site in the lattice interacts with z other sites. Use a mean-field approximation to derive an expression for the free energy. (Your answer should be left in terms of a solution to a transcendental equation.) In other words, replace the true nearest-neighbor interactions with an approximation in which every site interacts with every other site with an interaction strength rescaled appropriately. Is there a first-order (i.e. discontinuous in m) phase transition?

[Hint: it may be helpful to consider $\vec{m}$ of the form $(m, 0)$.]

- Solutions by Aram Harrow (August 2015)

  (a) $k_B \ln(q)$

  (b)

$$Z = \sum_{\sigma_1,\ldots,\sigma_N} M_{\sigma_1,\sigma_2} M_{\sigma_2,\sigma_3} \cdots M_{\sigma_{N-1},\sigma_N}, \tag{4}$$

where $M_{i,i} = e^{-\beta 3J/2}$ and $M_{i,j} = 1$ for $i \neq j$. Let I denote the $q \times q$ identity matrix and $u = \mathbf{1}/\sqrt{q}$, where $\mathbf{1}$ is the all-ones vector. Then

$$M = quu^T + (e^{-\beta 3J/2} - 1)I,$$

and so $M^N = (e^{-\beta 3J/2} + q - 1)^N uu^T + (e^{-\beta 3J/2} - 1)^N (I - uu^T)$. We compute

$$Z = \sum_{i,j} (M^N)_{i,j} = qu^T M^N u = q(e^{-\beta 3J/2} + q - 1)^N,$$

and so

$$F = -k_B T(\ln(q) + N \ln(e^{-\beta 3J/2} + q - 1)). \tag{5}$$

  (c) The inner product $\vec{s}_i \cdot \vec{s}_j$ is either 1 (when $i = j$) or $-1/2$ (when $i \neq j$). This yields the same Hamiltonian up to an overall additive constant.

  (d) We can replace the sum over $\langle i, j \rangle$ with $\frac{z}{N}$ times the sum over all pairs i, j. Then we get

$$E = -J\frac{z}{N} N^2 \vec{m} \cdot \vec{m} = -JzN|\vec{m}|^2.$$

If $\vec{m} = (m, 0)$ then the energy is $E = -JzNm^2$.

The entropy in this case can be determined from the populations N_1, N_2, N_3. Let $N_i = Np_i$. We can determine p_1, p_2, p_3 by solving the linear system of equations

$$p_1 - \frac{p_2 + p_3}{2} = m \tag{6}$$

$$p_2 - p_3 = 0 \tag{7}$$

$$p_1 + p_2 + p_3 = 1 \tag{8}$$

$$\tag{9}$$

which have solution $p_1 = \frac{1+2m}{3}$, $p_2 = p_3 = \frac{1-m}{3}$. The entropy is

$$S = Nk_B \sum_{i=1}^{3} p_i \ln(1/p_i) = \frac{Nk_B}{3}\left((1+2m)\ln\frac{3}{1+2m} + \frac{2(1-m)}{3}\ln\frac{3}{1-m}\right). \tag{10}$$

Thus the free energy is

$$F = E - TS = N\left(-Jzm^2 - \frac{k_B T}{3}\left((1+2m)\ln\frac{3}{1+2m} + \frac{2(1-m)}{3}\ln\frac{3}{1-m}\right)\right). \tag{11}$$

To calculate the derivative, observe that

$$\vec{\nabla} S(p_1, p_2, p_3) = \sum_{i=1}^{3} (-\ln(p_i) - 1) e_i \qquad \text{and} \qquad \frac{\partial \vec{p}}{\partial m} = \frac{1}{3} \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \tag{12}$$

so that $\frac{dS}{dm} = N k_B \frac{2}{3} \ln \frac{1-m}{1+2m}$.

The equilibrium value of m then satisfies

$$0 = \frac{1}{N} \frac{dF}{dm} = -2Jzm - \frac{2}{3} k_B T \ln \frac{1-m}{1+2m}, \tag{13}$$

or equivalently

$$\frac{1-m}{1+2m} = \exp(-3\beta J z m), \tag{14}$$

where $\beta \equiv 1/k_B T$.

There is always a solution at $m = 0$ corresponding to the disordered state. Whether there is a solution at $m > 0$ depends on the value of $\beta J z$. To see this, we evaluate the second derivative of F, which is proportional to

$$-3\beta J z - \frac{1}{1-m} + \frac{2}{1+2m}.$$

At $m = 0$ this is $1 - 3\beta J z$ which is negative for $\beta J z > 1/3$. When this occurs there is spontaneous order. This means that there is a first-order phase transition.

Statistical Mechanics 2: Magnetic susceptibility of a spinful Boltzmann gas

Consider a classical ideal gas of N spin-1/2 atoms moving in a container of volume V. In the presence of a weak external magnetic field H the energy of the nth such atom may be taken to be

$$E(\vec{p}_n, \sigma_n) = \frac{\vec{p}_n^{\,2}}{2m} - \gamma H \sigma_n \tag{1}$$

Here $\sigma_n = \pm 1$ describes the two possible spin orientations of the atom and $\vec{p}_n$ is the momentum of the atom. γ is a positive constant.

(a) (*2 pts*) Calculate the change in free energy due to the magnetic field.

(b) (*3 pts*) Calculate the average magnetization per atom $\langle M \rangle = \frac{1}{N}\gamma\langle\sum_n \sigma_n\rangle$.

(c) (*3 pts*) Calculate the variance in the magnetization $\langle M^2 \rangle - \langle M \rangle^2$.

(d) (*2 pts*) Use the above to calculate the magnetic susceptibility $\chi = \frac{d\langle M \rangle}{dH}$. How is the susceptibility related to the variance?

- Solutions by Senthil Todadri (August 2015)

(a) Partition function Z is given by

$$Z = \frac{1}{N!}V^N \left(\int \frac{d^3p}{2\pi\hbar} \sum_{\sigma=\pm1} e^{-\beta\left(\frac{\vec{p}^2}{2m} - \gamma\sigma H\right)} \right)^N \tag{2}$$

The translational and spin degrees of freedom are decoupled. Therefore the total partition function factorizes as

$$Z = Z_{trans} Z_s \tag{3}$$

with Z_{trans} independent of H (the partition function of an ideal gas of spinless atoms), and

$$Z_s = (2\cosh(\beta\gamma H))^N \tag{4}$$

The free energy

$$F = F_{trans} - \frac{N}{\beta}\ln(2\cosh(\beta\gamma H)) \tag{5}$$

(with $F_{trans} = -\frac{1}{\beta}\ln Z_{trans}$).

(b) The average magnetization per atom is obtained as

$$\langle M \rangle = -\frac{1}{N}\frac{\partial F}{\partial H} \tag{6}$$

This gives

$$\langle M \rangle = \gamma tanh(\beta\gamma H) \tag{7}$$

(c) The variance is obtained from the partition function through

$$\langle M^2 \rangle - \langle M \rangle^2 = \frac{1}{N^2\beta^2}\frac{\partial^2 Z}{\partial H^2} \tag{8}$$

$$= -\frac{1}{N^2\beta}\frac{\partial F}{\partial H} \tag{9}$$

$$= \frac{\gamma^2}{N} Sech^2(\beta\gamma H) \tag{10}$$

(d) Differentiatiing $\langle M \rangle$ with respect to field we get the magnetic susceptibility

$$\chi = \beta\gamma^2 Sech^2(\beta\gamma H) \tag{11}$$

We also have

$$\langle M^2 \rangle - \langle M \rangle^2 = \frac{k_B T\chi}{N} \tag{12}$$

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DOCTORAL GENERAL EXAMINATION

WRITTEN EXAM - QUANTUM MECHANICS — WITH SOLUTIONS

September 4, 2015

DURATION: 75 MINUTES

1. This examination has two problems. Read both problems carefully before making your choice. Submit ONLY one problem. IF YOU SUBMIT MORE THAN ONE PROBLEM FROM A SECTION, BOTH WILL BE GRADED, AND THE PROBLEM WITH THE LOWER SCORE WILL BE COUNTED.
2. If you decide at this time not to take this section of the exam, please inform the faculty proctor. ONCE YOU BEGIN THE EXAM, IT WILL BE COUNTED.
3. Calculators may not be used.
4. No books or reference materials may be used.

Quantum Mechanics 1: A spin-1/2 particle in a magnetic field

A spin-1/2 particle interacts with a magnetic field via the Hamiltonian

$$H = \vec{\sigma} \cdot \vec{B}, \qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$

(We absorb the other relevant constants into the definition of $\vec{B}$.)

(a) (*5 pts*) Find the eigenvalues and eigenvectors of H. You may find it convenient to write $\vec{B}$ as

$$\vec{B} = \begin{pmatrix} B\sin(\theta)\cos(\phi) \\ B\sin(\theta)\sin(\phi) \\ B\cos(\theta) \end{pmatrix}.$$

(b) (*5 pts*) Let $\vec{B} = B_0\hat{z}$. Suppose at time $t = 0$ the spin is initially pointing in the $+\hat{x}$ direction. After time $t = T$, the spin is measured along the $\hat{y}$ direction. What are the possible outcomes and what are their probabilities?

- Solutions by Aram Harrow (August 2015)

  (a) Observe that the Pauli matrices satisfy the multiplication rule $\{\sigma_i, \sigma_j\} = 2\delta_{i,j} I$, where $\{X, Y\} = XY + YX$ is the anticommutator and I is the 2×2 identity matrix. From this we calculate

  $$H^2 = \frac{1}{2}\{H, H\} = \frac{1}{2}\sum_{i,j=1}^{3} B_i B_j \{\sigma_i, \sigma_j\} = \sum_{i,j} B_i B_j \delta_{i,j} = B^2.$$

  Thus H has eigenvalues in the set $\{\pm B\}$. Since $\mathrm{tr} H = 0$, H must have one eigenvalue equal to B and one equal to $-B$. To calculate these we need to solve

  $$B \begin{pmatrix} \cos(\theta) & \sin(\theta)e^{-i\phi} \\ \sin(\theta)e^{i\phi} & -\cos(\theta) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \pm B \begin{pmatrix} x \\ y \end{pmatrix},$$

  for x, y. Assume without loss of generality that x is real (since we can always add an overall phase to make this true) and that $\sin(\theta) \neq 0$ (by continuity). Then from

  $$\pm x = \cos(\theta)x + \sin(\theta)e^{-i\phi}y,$$

  we find that $e^{-i\phi}y$ is also real. We can thus write $x = \cos(\alpha)$ and $y = \sin(\alpha)e^{i\phi}$. The eigenvalue equation then becomes

  $$\pm \begin{pmatrix} \cos(\alpha) \\ \sin(\alpha)e^{i\phi} \end{pmatrix} = \begin{pmatrix} \cos(\theta)\cos(\alpha) + \sin(\theta)\sin(\alpha) \\ (\sin(\theta)\cos(\alpha) - \cos(\theta)\sin(\alpha))e^{i\phi} \end{pmatrix} = \begin{pmatrix} \cos(\theta - \alpha) \\ \sin(\theta - \alpha)e^{i\phi} \end{pmatrix}.$$

  The $+B$ eigenstate then has $\alpha = \theta/2$ while the $-B$ eigenstate has $\alpha = \pi + \theta/2$, and so the corresponding eigenvectors are

  $$|{+B}\rangle = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)e^{i\phi} \end{pmatrix} \qquad \text{and} \qquad |{-B}\rangle = \begin{pmatrix} -\sin(\theta/2) \\ \cos(\theta/2)e^{i\phi} \end{pmatrix} \tag{2}$$

  (b) The state after time T is

  $$e^{-iHT/\hbar}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-iBT/\hbar} \\ e^{iBT/\hbar} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\omega_0 T} \\ e^{i\omega_0 T} \end{pmatrix},$$

  where we have defined $\omega_0 = B/\hbar$. The possible outcomes after a $\hat{y}$ measurement are $\pm\hat{y}$, corresponding to vectors $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm i \end{pmatrix}$. The corresponding probabilities are

  $$\Pr[\pm\hat{y}] = \left|\frac{e^{-i\omega_0 T} \pm i e^{i\omega_0 T}}{2}\right|^2.$$

  These cases can be separately calculated as

  $$\Pr[+\hat{y}] = \left|\frac{e^{-i\omega_0 T + i\frac{\pi}{4}} + e^{i\omega_0 T - i\frac{\pi}{4}}}{2}\right|^2 = \sin^2(\omega_0 T - \pi/4) \tag{3}$$

  $$\Pr[-\hat{y}] = \left|\frac{e^{-i\omega_0 T - i\frac{\pi}{4}} + e^{i\omega_0 T + i\frac{\pi}{4}}}{2}\right|^2 = \sin^2(\omega_0 T + \pi/4) \tag{4}$$

Quantum Mechanics 2: Two interacting fermions

Consider two identical fermions of mass m interacting with each other through an attractive harmonic potential. The Hamiltonian is

$$H = \frac{\mathbf{p}_1^2}{2m} + \frac{\mathbf{p}_2^2}{2m} + \frac{k}{2}\left(\mathbf{x}_1 - \mathbf{x}_2\right)^2 \tag{1}$$

$\mathbf{x}_{1,2}$ are the coordinates of the two fermions, and $\mathbf{p}_{1,2}$ are the conjugate momenta. For simplicity assume that the spins of both fermions are polarized in the same direction (e.g, in the $\uparrow$ direction) so that we may ignore the spin degree of freedom.

(a) (*2 pts*) State the restriction imposed by Fermi statistics on acceptable wave functions $\psi(\mathbf{x}_1, \mathbf{x}_2)$ of this system.

(b) (*2 pts*) Rewrite H in terms of center-of-mass and relative coordinates.

(c) (*3 pts*) Ignoring the restriction imposed by Fermi statistics what is the ground state energy? What is the energy of the first excited bound state?

(d) (*3 pts*) Including the effects of Fermi statistics what is the ground state energy? What is the degeneracy of the ground state?

- Solutions by Senthil Todadri (August 2015)

  (a) The wave function must be antisymmetric under exchange: $\psi(\mathbf{x}_1, \mathbf{x}_2) = -\psi(\mathbf{x}_2, \mathbf{x}_1)$.

  (b) Introduce center-of-mass $\mathbf{R} = \frac{\mathbf{x}_1+\mathbf{x}_2}{2}$, and relative coordinates $\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2$. The corresponding conjugate momenta are $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2$ and $\mathbf{p} = \frac{\mathbf{p}_1-\mathbf{p}_2}{2}$. In terms of these the Hamiltonian is

  $$H = \frac{\mathbf{P}^2}{4m} + \frac{\mathbf{p}^2}{m} + \frac{k\mathbf{x}^2}{2} \quad (2)$$

  (c) As $\mathbf{P}$ commutes with the rest of the Hamiltonian we can set fix it to determine the eigen-energies. The ground state and the lowest energy excited bound states will have $\mathbf{P} = 0$. The relative motion is described by a $3d$ simple harmonic oscillator at a frequency $\omega = \sqrt{\frac{2k}{m}}$.

  Therefore ground state energy is $\frac{3}{2}\hbar\omega$ and first excited bound state is at $\frac{5}{2}\hbar\omega$. The ground state is singly degenerate and the excited bound state is triply degenerate.

  (d) The antisymmetry condition means that the orbital angular momentum L of the relative coordinate must be odd. In the ground state of the $3d$ oscillator, $L = 0$ and hence this is not acceptable once we impose Fermi statistics. The first excited bound states have $L = 1$, and hence they are acceptable states. Thus for fermions, the ground state energy is $\frac{5}{2}\hbar\omega$, and the ground state is three fold degenerate.