# Computing High-Precision Lewis Weights

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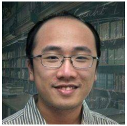

# Problem Statement

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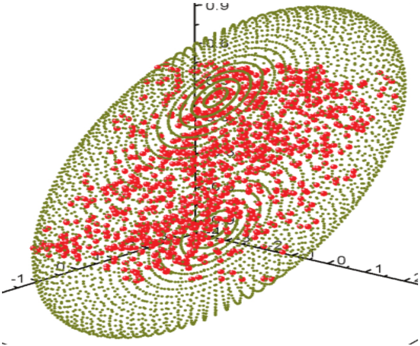

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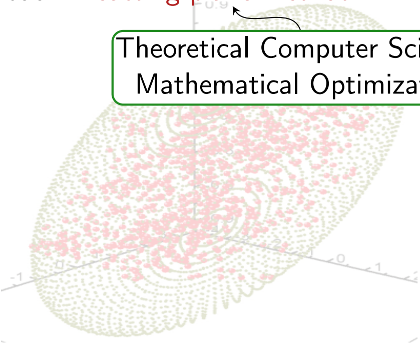
- Algorithmic tool in cutting-plane method

Theoretical Computer Science, Mathematical Optimization



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Statistics,
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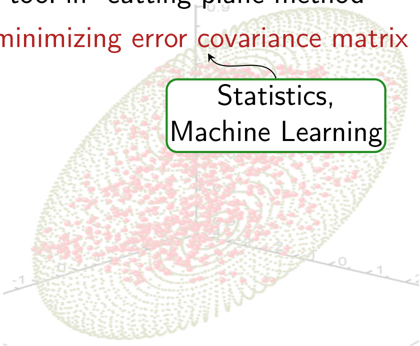

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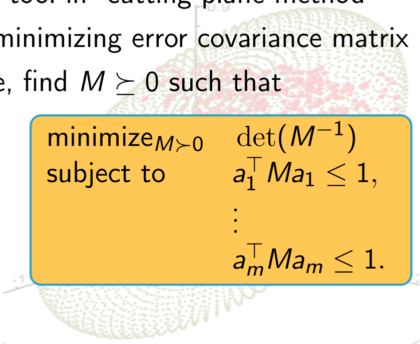
- To compute, find $M \succeq 0$ such that

$$\begin{array}{ll} \text{minimize}_{M \succ 0} & \det(M^{-1}) \\ \text{subject to} & a_1^\top M a_1 \leq 1, \\ & \vdots \\ & a_m^\top M a_m \leq 1. \end{array}$$



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- Optimal ellipsoid is $M^{-1} = \sum_{i=1}^m \overline{w}_i a_i a_i^\top$, where

$$a_i^\top (A^\top \mathrm{diag}(\overline{w}) A)^{-1} a_i = 1.$$

# More Generally: Lewis Ellipsoid

Find the Lewis Ellipsoid $M \succeq 0$ that satisfies

$$\begin{array}{ll} \text{minimize}_{M \succ 0} & \det(M^{-1}) \\ \text{subject to} & \sum_{i=1}^{m} (a_i^\top M a_i)^{p/2} \leq 1 \end{array} .$$

# More Generally: Lewis Ellipsoid

$$\begin{array}{ll} \text{minimize}_{M \succ 0} & \det(M^{-1}) \\ \text{subject to} & a_i^\top M a_i \leq 1, \forall i \end{array}$$

Find the Lewis Ellipsoid $M \succeq 0$ that satisfies

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$p = \infty$ gives John Ellipsoid

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Existence proved by D.R. Lewis in 1978

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  - sampling important rows of data matrix,
  - computing the Lee-Sidford self-concordant barrier

Mathematical Optimization

# Optimization Perspective

Lewis weights are the solution to the convex program

$\alpha = \frac{2}{p-2}$

$$\operatorname{argmin}_{w \geq 0} \quad -\log\det(A^\top \operatorname{diag}(w) A) + \frac{1}{1+\alpha} 1^\top w^{1+\alpha}.$$

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sampling

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- Runtime measured in number of leverage score computations
- Concretely:

Compute $\ell_p$ Lewis weights with $\mathrm{poly}\log(1/\varepsilon)$ leverage scores.

# Prior Work on Computing Lewis Weights

# Contrasting Our Results with Previous Work

| Year | Authors | Range of $p$ | # Leverage Score Computations | Total Run Time |
|---|---|---|---|---|
| 2015 | Cohen, Peng | $p \in (0, 4)$ | $O(\frac{1}{1-\lvert 1-p/2\rvert} \log(\frac{\log m}{\varepsilon}))$ | $O(\frac{1}{\lvert 1-p/2\rvert} mn^2 \log(\frac{\log m}{\varepsilon}))$ |
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| 2021 | Our result | $p \geq 4$ | $O(p^3 \log(\frac{m}{\varepsilon}))$ | $O(p^3 mn^2 \log(\frac{m}{\varepsilon}))$ |

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- Previously: high-precision algorithm only for $0 \leq p < 4$
- We provide a high-precision algorithm for $p \geq 4$.
- Therefore, there now exists a high-precision algorithm for all $p > 0$.

# Prior Work: Cohen-Peng'15

- Repeatedly, $w_i \leftarrow (a_i^\top (A^\top \mathrm{diag}(w)^{1-2/p} A)^{-1} a_i)^{p/2}$ for all $i$.

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For $p \geq 4$, solve the convex program

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- Other approaches: mirror descent, homotopy method, ...

# Our Contributions

# Our Technical Outline

We compute Lewis weights by solving:

$$\text{argmin}_{w \geq 0} \quad -\log\det(A^\top \text{diag}(w) A) + \frac{1}{1+\alpha} 1^\top w^{1+\alpha}.$$

$\alpha = \frac{2}{p-2}$

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Leverages upper & lower quadratics

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- Standard high-accuracy algorithms don't apply
- Identify a "rounding" condition that allows geometric decrease in error
- Maintain the rounding condition in constant number of steps

# Our Algorithm

1. Initialize $w^{(0)} = n/m$.
2. For $k = 1, 2, 3, \ldots, T$ iterations, do:
   - $\widetilde{w}^{(k)} \leftarrow \textbf{Round}(w^{(k-1)})$ // rounding condition
   - $w^{(k)} \leftarrow \textbf{Descent}(\widetilde{w}^{(k)}, [m], 1)$ //make progress
3. Set $w_R \leftarrow \textbf{Round}(w^T)$
4. Return $\widehat{w}$ where $\widehat{w}_i \leftarrow (a_i^\top (\mathrm{AW}_R\mathrm{A})^{-1} a_i)^{1/\alpha}$.

# Our First Technical Building Block: **Descent**

$$\rho_i(w) = \frac{a_i^\top (A^\top \mathrm{diag}(w) A)^{-1} a_i}{w_i^\alpha}$$

The descent step:

$$[\mathrm{Descent}(w, C, \eta)]_i := w_i \left[1 + \eta_i \cdot \frac{\rho_i(w) - 1}{\rho_i(w) + 1}\right].$$

for $i \in C \subseteq \{1, 2, \ldots, m\}$

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- Inspired by Newton step
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- Key properties:
  - $C = [m]$ and "rounding" condition satisfied: fast progress
  - Otherwise, fixes the "rounding" condition

# Our Second Technical Building Block: **Rounding**

The rounding condition:

$\sigma_i(w) = w_i \cdot a_i^\top (A^\top WA)^{-1} a_i$

$$\max_{i \in [m]} \rho_i(w) \le 1 + \alpha, \text{ where } \rho_i(w) = \frac{\sigma_i(w)}{w_i^{1+\alpha}}.$$

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Rounding Algorithm:

1. while $C = \{i : \rho_i(w) > 1+\alpha\} \neq \emptyset$,
   - $w \leftarrow \textbf{Descent}(w, C, \frac{1}{\alpha})$
2. Return $w$

# Analysis of Algorithm: **Descent** Step

Want to bound iteration complexity of **Descent** steps, where

$$\mathbf{Descent}(w, C, \eta)]_i := w_i \left[1 + \eta_i \cdot \frac{\rho_i(w)-1}{\rho_i(w)+1}\right]$$

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Proof Idea: Upper bound on sub-optimality, lower bound on progress.

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**Lemma 2: Descent Step Convergence**

Each **Descent** step decreases the objective, and the number of **Descent** steps is $O(\alpha^{-1}\log(m/\varepsilon))$.

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Proof Sketch: Combine **Round**'s termination condition $\rho_{\max}(w) \leq 1 + \alpha$ with the maximum increase in $\rho_{\max}(w)$ per `while` loop .

# Round Step Complexity of `while` Loop: Bounding $\rho_{\max}(w)$

**Lemma 4: Round change in $\rho_{\max}(w)$**

Let $w^+$ be the state of $w$ after one `while` loop of **Round**. Then,

$$\rho_{\max}(w^+) \le \left(1 + \frac{\alpha}{1+\alpha}\right)^{-\alpha} \rho_{\max}(w).$$

$$w_i^+ = w_i + \frac{w_i}{3}\left(\frac{\rho_i(w)-1}{\rho_i(w)+1}\right) \quad \text{//Descent on } i \in C$$

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Combine with $w_i = w_i^+$ for all $i \notin C$ implies $w^+ \ge w$. Hence:

$$\rho_i(w^+) = [w_i^+]^{-\alpha}\left[A(A^\top \mathrm{diag}(w^+)A)^{-1}A^\top\right]_{ii} \le \left[1 + \frac{\alpha}{3(2+\alpha)}\right]^{-\alpha} \rho_i(w)$$

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| Therefore, over $\mathcal{D}$ iterations: | $\mathcal{R} := \mathcal{O}(\mathcal{D} \cdot \alpha^{-2} \log(m/n))$ |

# Our Contribution

**Lemma 5: Our Main Result (Informal)**

Given a full-rank $A \in \mathbb{R}^{m \times n}$ and $p \geq 4$, we compute $\varepsilon$-approximate $\ell_p$ Lewis weights in $O(p^3 \log(mp/\varepsilon))$ oracle calls.

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- Our algorithm has both a parallel and a sequential version.
- Computing gradient of Lee-Sidford barrier to high-precision in polylogarithmic depth.

Thank You!