Complete solutions to the problems

Only for people who actually care about the proofs

The IMO Shortlist

41st Mystery Hunt 2021

§1 Algebra

Solution A1

This is a quadratic equation in y, so we may simply expand to get

$$9y^2 + (24x - 12(4x+1))y + 16x^2 + 4(4x+1) = 0$$

which factors as

$$(3y - (4x+2))^2 = 0.$$

Hence, the solution is $y = \frac{4x+2}{3}$.

Solution A2

We let $A > 0$ denote the desired expression. Square both sides to obtain

$$\begin{aligned} A^2 &= \left(20x + 3 + 4\sqrt{15x}\right) + \left(20x + 3 - 4\sqrt{15x}\right) + 2\sqrt{(20x+3)^2 - (4\sqrt{15}x)^2} \\ &= 40x + 6 + 2\sqrt{400x^2 - 120x + 9} \\ &= 40x + 6 + 2(20x - 3) = 80x. \end{aligned}$$

Therefore, $A = \sqrt{80x}$.

Solution A3

Let $g(x) = \frac{f(x)+1}{3}$. Then the condition means we have the convolution

$$g * \mathbf{1} = \mathrm{id}$$

where $\mathbf{1}$ is the constant function 1. This means that g coincides with the Euler phi function φ. Hence, it follows that

$$f(x) = 3\varphi(x) - 1.$$

Solution A4

Start with the identity

$$\sum_{n\geq 0} T^n = \frac{1}{1-T}$$

valid for $|T| < 1$. Differentiate both sides to obtain $\sum_{n\geq 0} nT^{n-1} = \frac{1}{(1-T)^2}$ or equivalently

$$\sum_{n\geq 0} nT^n = \frac{T}{(1-T)^2}.$$

Consequently, if we choose $T = \frac{1}{1+x^{-1/2}}$, we obtain

$$\begin{aligned}
\sum_{n\geq 0} n\left(1+\frac{1}{\sqrt{x}}\right)^{-n} &= \frac{\frac{1}{1+\frac{1}{\sqrt{x}}}}{\left(1-\frac{1}{1+\frac{1}{\sqrt{x}}}\right)^2} \\
&= \frac{1+\frac{1}{\sqrt{x}}}{\left(\left(1+\frac{1}{\sqrt{x}}\right)-1\right)^2} \\
&= x+\sqrt{x}.
\end{aligned}$$

Solution A5

This problem is taken from Putnam 2008 B3 which has the following form:

> **Claim —** The largest possible radius of a circle inside an n-dimensional hypercube of side length $s > 0$ is exactly $\frac{s}{2}\sqrt{\frac{n}{2}}$.

Proof. By scaling, we assume without loss of generality that $s = 2$. By symmetry, we may assume the center is zero.

Parametrize the circle as

$$\mathbf{x}(t) = (\mathbf{a}\cos t + \mathbf{b}\sin t)$$

where vectors

$$\begin{aligned}
\mathbf{a} &= (a_1, \dots, a_n) \\
\mathbf{b} &= (b_1, \dots, b_n)
\end{aligned}$$

are orthogonal of the same length r. We require that every component is contained inside $[-1, 1]$ across all $t \in \mathbb{R}$.

For the upper bound, note that we thus require $\sqrt{a_i^2 + b_i^2} \leq 1$ for every i, whence squaring and summing gives $n \geq \sum_i (a_i^2 + b_i^2) = 2r^2$.

For the lower bound, we need two constructions.

- For $n \geq 2$ odd $\mathbf{a} = (1, \ldots, 1, 0, \ldots, 0)$ and $\mathbf{b} = (0, \ldots, 0, 1, \ldots, 1)$.
- For $n \geq 3$ odd, we can set

$$\mathbf{a} = \left(\frac{\sqrt{3}}{2}, \frac{-\sqrt{3}}{2}, 0\right)$$
$$\mathbf{b} = \left(\frac{1}{2}, \frac{1}{2}, -1\right)$$

  for $n = 3$ and then inductively add two components. $\square$

Applying this to the given problem, we find the necessary and sufficient condition d is that

$$\sqrt{x} \leq \frac{\pi}{2}\sqrt{\frac{d}{2}}.$$

Put another way, we have $d \geq \frac{8}{\pi^2}x$. Since d must be an integer, we conclude

$$d \geq \left\lceil \frac{8}{\pi^2}x \right\rceil.$$

Solution A6

We rewrite the given equation as

$$2f(x+y) - 1 = 4f(x)f(y) - 2f(x) - 2f(y) + 1 = (2f(x) - 1)(2f(y) - 1).$$

Letting $g(x) = 2f(x) - 1$, we find $g(x+y) = g(x)g(y)$.

Note that if any value of g is equal to zero, then the function g is identically zero, which is not a valid solution since we assumed the function f was strictly increasing. So in what follows we assume g is never zero.

Since $g(x) = g(x/2)^2 > 0$ for any x, it follows that $\log g$ is a well-defined additive function (meaning $\log g(x+y) = \log g(x) + \log g(y)$) which is also strictly increasing. It is a property of Cauchy's equation that this implies $\log g$ is a linear function. Consequently, we conclude that g is an exponential function.

Putting this all together, we derive that

$$f(x) = \frac{c^x + 1}{2}$$

for some constant c. Since we need $f(24) = 5$, it follows that the correct constant is $c = 9^{1/24}$. In other words,

$$f(x) = \frac{9^{x/24} + 1}{2}$$

is the only solution to the functional equation.

§2 Combinatorics

Solution C1

There are $\binom{2n}{n}$ ways to choose $A \cup B$ after which there are 2^n ways to split $A \cup B$ into sets A and B. In other words, we find

$$M = 2^n \binom{2n}{n}.$$

By Legendre's formula, the exponent of 2 in $n!$ is $n - s_2(n)$, while the exponent of 2 in $(2n)!$ is $2n - s_2(2n) = 2n - s_2(n)$, where $s_2(n)$ is the number of 1's in the binary representation of n. So the exponent of 2 in $\binom{2n}{n}$ is exactly $[2n - s_2(n)] - 2[n - s_2(n)] = s_2(n)$.

From this we conclude that

$$e = n + s_2(n)$$

is the final answer.

Solution C2

This problem is adapted from USAMO 2009 problem 2 and the answer is $2\lceil n/2 \rceil$.

By shifting, we will work on the analogous problem where we want a subset of $\{-n, -(n-1), \dots, 0, \dots, n\}$ and we want no three elements to have sum equal to zero.

To show $2\lceil n/2 \rceil$ is achievable it now suffices to take all odd numbers.

We now turn our attention to showing this is best possible. To prove this is maximal, it suffices to show it for n even; we do so by induction on even $n \geq 2$ with the base case being trivial. Letting A be the subset, we consider three cases:

1. If $|A \cap \{-n, -n+1, n-1, n\}| \leq 2$, then by the hypothesis for $n-2$ we are done.
2. If both $n \in A$ and $-n \in A$, then there can be at most $n-2$ elements in $A \setminus \{\pm n\}$, one from each of the pairs $(1, n-1)$, $(2, n-2)$, $\dots$ and their negations.
3. If $n, n-1, -n+1 \in A$ and $-n \notin A$, then at most $n-3$ more can be added, one from each of $(1, n-2)$, $(2, n-3)$, $\dots$ and $(-2, -n+2)$, $(-3, -n+3)$, $\dots$. (In particular $-1 \notin A$. Analogous case for $-A$ if $n \notin A$.)

Thus in all cases, $|A| \leq n$ as needed.

Solution C3

This problem is based on China TST 2018, test 2, problem 3.

The answer to the given problem is $\frac{n+2014}{\gcd(n,2014)}$. In general, we replace 2014 by m and show the answer $\frac{n+m}{\gcd(m,n)}$.

Note that if $d = \gcd(m,n) > 1$, we could focus only on the numbers which are $0 \pmod d$, which is the same process all numbers scaled by d and the pair (m,n) replaced by the pair $(m/d, n/d)$. The same logic applies to numbers which are $1 \pmod d$, etc.

Thus we may scale down by a factor of d without any loss of generality. So in what follows, we will always assume $\gcd(m, n) = 1$, and show the answer is $m + n$.

To see that the operation could go on indefinitely, we notice that writing $\{1, \dots, m\}$ and $\{1, \dots, n\}$ is sufficient.

Now we prove that fewer than $m + n$ numbers does not suffice. In fact it suffices to work modulo $m + n$. We consider the finite simple undirected graph G defined on the vertex set $V = \mathbb{Z}/(m + n)$ with edges $x \to x + m$ and $x \to x + n$ for each $x \in \mathbb{Z}/(m + n)$. (These are equivalent to $x \to x - n$ and $x \to x - m$ modulo $m + n$, respectively.) Since $\gcd(m, n) = 1$, this graph is connected and has $m + n$ edges (actually, it is isomorphic to a cycle on $m + n$ vertices).

For every number on the board, we place a chip at the corresponding residue in V and then observe the given operation corresponds to chip-firing. It is a standard lemma from chip-firing that the number of chips needed for chip-firing to continue indefinitely is at least the number of edges of G, which is exactly $m + n$.

Solution C4

This problem is adapted from USA Team Selection Test for IMO 2015, problem 5.

First, we eliminate the distractions of coloring the vertices from the problem:

> **Claim** — The vertices are pairwise distinct colors. Moreover, no color used on an edge can be used on a vertex.

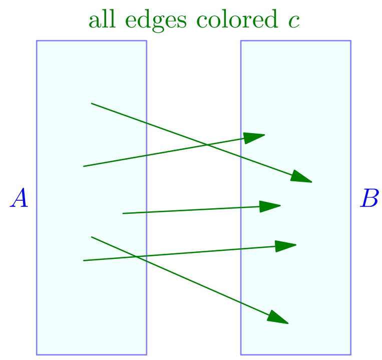
Proof. It suffices to verify this condition among any sub-tournament with only three vertices (since given an edge e and a vertex v we can find three vertices containing both endpoints of e and the vertex v).

Now among any three vertices there is always a way to label them a, b, c such that $a \to b$ and $b \to c$, and so the conclusions of the claim follows. $\square$

Now suppose there are χ colors represented among the edges.

> **Claim** — The minimum possible value of χ is $\lceil \log_2 n \rceil$.

Proof. First, we prove by induction on n that $\chi \geq \log_2 n$ for any coloring and any tournament. The base case $n = 1$ is obvious. Now given any tournament, consider any used color c. Then it should be possible to divide the tournament into two subsets A and B such that all c-colored edges point from A to B (for example by letting A be all vertices which are the starting point of a c-edge).



One of A and B has size at least $n/2$, say A. Since A has no c edges, and uses at least $\log_2 |A|$ colors other than c, we get

$$\chi \geq 1 + \log_2(n/2) = \log_2 n$$

completing the induction.

One can read the construction off from the argument above, but here is a concrete description. For each integer n, consider the tournament whose vertices are the binary representations of $S = \{0, \dots, n-1\}$. Instantiate colors c_1, c_2, ... Then for $v, w \in S$, we look at the smallest order bit for which they differ; say the kth one. If v has a zero in the kth bit, and w has a one in the kth bit, we draw $v \to w$. Moreover we color the edge with color c_k. This works and uses at most $\lceil \log_2 n \rceil$ colors. $\square$

Hence, the final answer to the question is $n + \lceil \log_2 n \rceil$.

Solution C5

For $n > 6$ we claim the answer is $n + 6$. This is adapted from Tournament of Towns, Spring 2016, Problem J-A7.

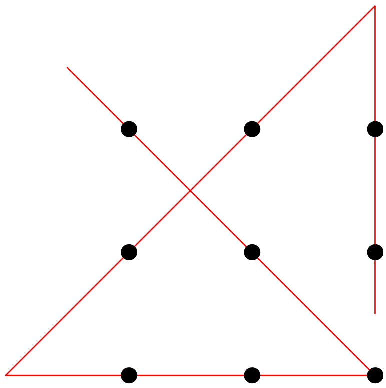
This is a consequence of Turán's theorem. View the cards as vertices of a graph G, and the pointed cards as edges of the graph G. Happiness is guaranteed if and only if it is impossible to find an *independent set* with at least $n/2$ cards.

Turán's theorem then implies that the graph G consisting of $\frac{n-6}{2}$ copies of K_2 and 2 copies of K_3, which obviously has independence number $\frac{n-6}{2} + 2 = \frac{n}{2} - 1$, is best possible.

Solution C6

Let $\ell = 2n$. We claim that $\ell \geq 2m - 2$ for $m \geq 3$ and that this is best possible.

We will only deal with the case $m \geq 3$. For the construction, the case $m = 3$ is a famous brainteaser (shown below), and then one can inductively add a vertical/horizontal line in order to to get $\ell = 2m - 2$ for any $m \geq 4$.



Now, we show $\ell \geq 2m-2$. Assume there are $m-a$ horizontal lines and $m-b$ vertical lines. We first dispense of some edge cases:

- If $a \leq 0$, then there are at least m horizontal lines, which must be joined by at least $m-1$ other lines, so $\ell \geq 2m-1$.
- If $a = 1$, then there is some row of points is untouched by the horizontal lines, and at least m non-horizontal lines are needed to pass through these. So $\ell \geq (m-1)+m = 2m-1$.

The cases $b \leq 0$ and $b = 1$ are handled analogously. Hence in what follows assume $\min(a,b) \geq 2$.

Then we have an $a \times b$ sub-grid of points (perhaps not evenly spaced, but still rectangular) not touched by any horizontal or vertical lines. Consider the boundary of this grid, which has $2a+2b-4$ points. Each line passes through at most 2 points on this boundary. So we need at least $a+b-2$ lines not horizontal or vertical. In conclusion,

$$\ell \geq (a+b-2)+(m-a)+(m-b) = 2m-2.$$

Since $\ell = 2n$, we get $n \geq m-1$ as needed.

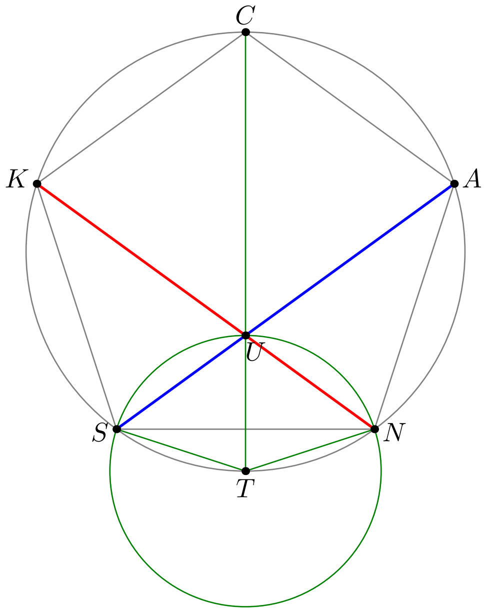
Put another way, we have $m \leq n+1$ as the answer.

§3 Geometry

Solution G1

The hidden collinearity is that U, N, K are collinear, which we prove now.

It is a classical lemma that since T is the arc midpoint of $\widehat{SN}$, the point U is the incenter of $\triangle CSN$. Consequently, since A is the arc midpoint of $\widehat{SK}$, it follows that ray SA is the angle bisector of $\angle CS$, so it passes through the point U.

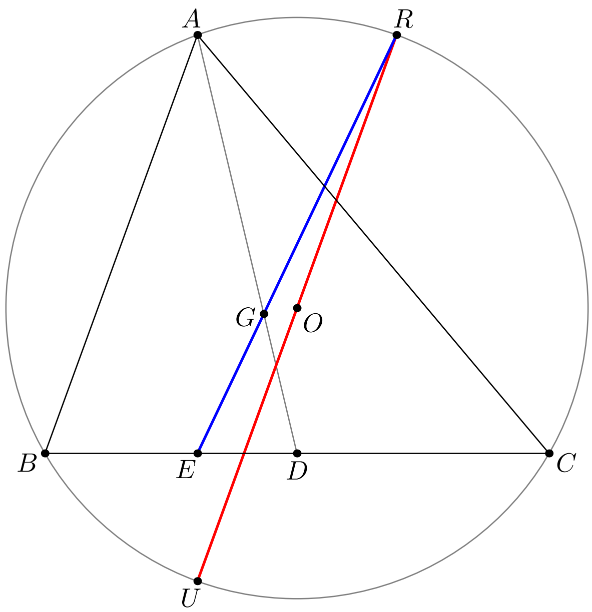


(For a symmetric reason, the point U also lies on line SA.)

Solution G2

The hidden collinearity is that G, E, R are collinear, which we prove now.

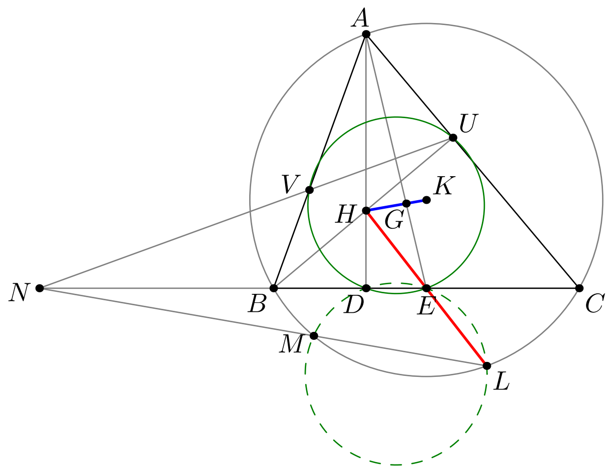
Evidently, the point R is the one such that $ABCR$ is an isosceles trapezoid. There is a negative homothety (with ratio -2) which sends the nine-point circle to the $ABCR$, which maps D to A and E to R.



(As for the originally posed problem, the point U is the reflection of the orthocenter over line BC, which is known to be the antipode of R.)

Solution G3

The hidden collinearity in the picture is that H, E, L are collinear, which we prove now.



Consider the circumcircles of $UVDE$ (nine-point circle), ABC, and the triangle MDE. The pairwise radical axii must be concurrent; note that line DE is one radical axis. On the other hand, since $BVUC$ is a cyclic quadrilateral, we know

$$ND \cdot NE = NU \cdot NV = NB \cdot NC$$

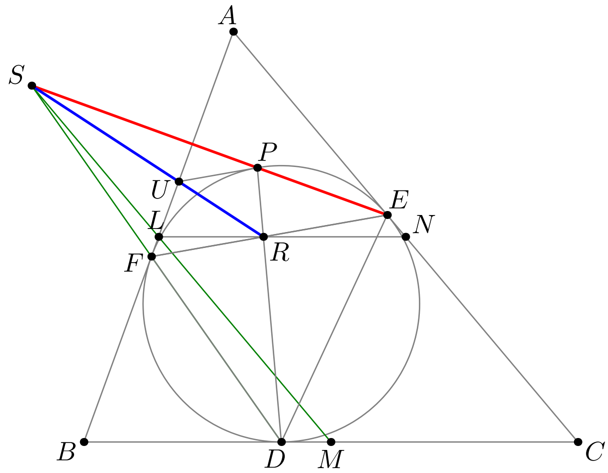
so the point N is the radical center, since it also lies on line DE.

In other words, the line MN must pass through the second intersection point of (MDE) and (ABC). Hence $MDEL$ is cyclic. But since $\angle AMN = 90°$, it follows L must actually coincide with the antipode of the point A. It's well known the reflection of H across E is the desired antipode, so we're done.

(As for the originally proposed question, K is now the circumcenter of ABC so the line HKG is simply the Euler line.)

Solution G4

The hidden collinearity in the picture is that E, S, P are collinear.

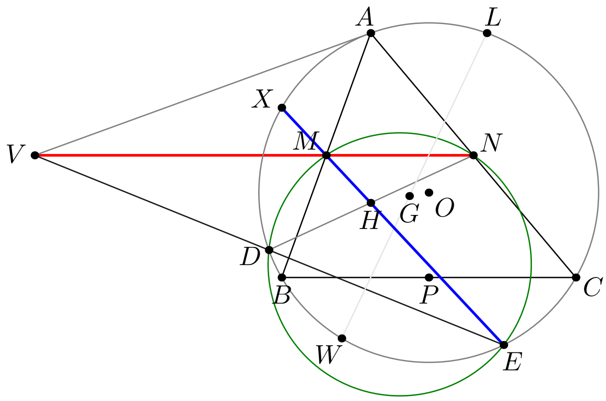


We will need that the Feurerbach hyperbola is the rectangular circumhyperbola of triangle ABC which passes through the point I, and that the center of this hyperbola is the Feurerbach point P. Then the conclusion follows by the First Fontené Theorem: the points $R = EF \cap NL$, $S = FD \cap LM$, $T = DE \cap MN$ (not shown) should have lines DR, ES, FT meeting at the point P.

(For the actually posed question, note first by Brokard's theorem on $PFDE$ that the points $U = PP \cap FF$, $R = PD \cap EF$, $S = PE \cap DF$ are collinear. Incidentally, the point $C = DD \cap EE$ lies on this line too.)

Solution G5

The hidden collinearity in the picture is that V, M, N are collinear, which we prove now. (Technically speaking, OGH and COX are also collinear, but there is no country with code using those triples of letters.)



To prove this, note that a negative inversion at H carrying the nine-point circle to the circumcircle will map M to E and N to D, so that the points M, N, E, D are cyclic. In addition, the circumcircle of AMN is tangent to the circumcircle of ABC at the point A. So considering the radical axis of the circles AMN, ABC, $MNDE$ shows that the tangent at A, the line DE, and the line MN concur — at V. Hence V, M, N are collinear.

(For the originally suggested problem, one can note that VW is tangent to the circumcircle of ABC and then, following the approach of IMO Shortlist 2011 G4, show that L is the point for which $ABCL$ is an isosceles trapezoid. The contrived angle condition that X is the C-antipode, as needed.)

§4 Number Theory

Solution NT1

This is a classical variant of the Frobenius coin problem. We need $\gcd(m,n)=1$ or else any integer relatively prime to $\gcd(m,n)$ is not expressible.

It is known that in when $\gcd(m,n)=1$ then the number of integers which can't be formed is
$$\frac{mn-m-n}{2}.$$
So we wish to solve
$$\frac{mn-m-n}{2}=4161 \iff (m-1)(n-1)=mn-m-n+1=8322.$$
Since $8322=2\cdot 3\cdot 19\cdot 73$, the requested pair is $(m-1,n-1)=(73,114)$ or
$$(m,n)=(74,115).$$

Solution NT2

The condition rewrites as
$$(7m+2n)^2+(3m+n)^2=193.$$
The number 193 is prime and can be expressed as the sum of two squares in exactly one way, $193=12^2+7^2$. So we want either $7m+2n=\pm 12$ and $3m+n=\pm 7$, or $7m+2n=\pm 7$ and $3m+n=\pm 12$.

We exhaust all cases below:

System of equations		Solution
$7m+2n=\pm 12$	$3m+n=\pm 7$	$(m,n)=(\mp 2,\pm 13)$
$7m+2n=\pm 12$	$3m+n=\mp 7$	$(m,n)=(\pm 26,\mp 85)$
$7m+2n=\pm 7$	$3m+n=\pm 12$	$(m,n)=(\mp 17,\pm 63)$
$7m+2n=\pm 7$	$3m+n=\mp 12$	$(m,n)=(\pm 31,\mp 105)$.

For n to be as large as possible, we want the pair $(-31,105)$.

Solution NT3

Obviously we need $m \geq 0$ or the LHS is not an integer. When $m = 0$ or $m = 1$ there is no solution so we assume $m \geq 2$.

First, taking modulo 8, we have $3^m \equiv (2n-3)^2 \equiv 1 \pmod 8$, integer m must be even. When considering $8(m-1)^2 = 3^m - (2n-3)^2$, as both terms are positive we conclude that both factors are positive, i.e. that $3^{m/2} > |2n-3|$. Therefore, we obtain the crude estimate

$$8(m-1)^2 \geq 3^m - (3^{m/2}-1)^2 = 2 \cdot 3^{m/2} - 1.$$

which in fact can only hold for $m \leq 10$.

We now check all the cases:

m	$3^m - 8(m-1)^2$	$2n-3$	n
$m = 2$	1	1	$n = 1$ or $n = 2$
$m = 4$	9	3	$n = 0$ or $n = 3$
$m = 6$	529	23	$n = -10$ or $n = 13$
$m = 8$	6169	n/a	n/a
$m = 10$	58401	n/a	n/a.

This gives $(6, -10)$ as the pair with $m - n$ as large as possible.

Solution NT4

By AM-GM, we have

$$f(\pi) \geq \sqrt[3]{9!} \approx 213.98.$$

So we have $n \geq 214$ and $9m \geq 216$. Both can be achieved:

$$1 \cdot 8 \cdot 9 + 2 \cdot 5 \cdot 7 + 3 \cdot 4 \cdot 6 = 214$$
$$1 \cdot 8 \cdot 9 + 2 \cdot 6 \cdot 7 + 3 \cdot 4 \cdot 5 = 216.$$

Hence the answer $(24, 214)$.