Closure Properties of D_{2p} in Finite Groups

Po-Ling Loh

under the direction of
Dr. Michael Aschbacher
California Institute of Technology

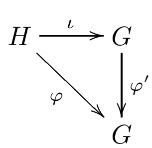
Abstract

We discuss the generalized version of a question posed by the topologist E. Farjoun about closed embeddings of a finite group H in a finite group G. We study the case $H \cong D_{2p}$ for p an odd prime and determine a sufficient condition for $H = G$ when H is closed in G.

1 Introduction

Group theory is of importance in many fields of mathematics, including topology. The problem discussed in this paper is related to a problem posed by the topologist Farjoun.

Let H and G be objects in a category $\mathbf{C}$, and let $\iota \in \mathrm{Mor}(H, G)$. Farjoun studies morphisms ι such that for each $\varphi \in \mathrm{Mor}(H, G)$, there exists a unique $\varphi' \in \mathrm{Mor}(G, G)$ such that $\iota\varphi' = \varphi$. In other words, the following diagram commutes.



One special case arises when $\mathbf{C}$ is the category of groups and group homomorphisms, and ι is an inclusion map. We denote by $\mathrm{Hom}(H, G)$ the set of homomorphisms from H to G and write $\mathrm{End}(G) = \mathrm{Hom}(G, G)$. We say H is *closed* in G if $H \leq G$ and each $\varphi \in \mathrm{Hom}(H, G)$ extends uniquely to some $\varphi' \in \mathrm{End}(G)$.

Farjoun studies the properties of H that are preserved when H is closed in G. In particular, he asks the following question:

Question. *If H is finite, closed, and nilpotent, is $H = G$?*

It is known (cf. Remark 3.7) that if G is nilpotent and H is closed in G, then $H = G$. Therefore, Farjoun's question can be interpreted as asking whether nilpotence is preserved under the closure of finite groups. For more material related to Farjoun's question, we refer the reader to [3], [5], and [6].

In the discussion that follows, we will remove the condition that H be nilpotent and analyze general closed embeddings of finite groups in finite groups. In Section 2, we will consider the case when H and G are simple. Since H and G have only two normal subgroups apiece, we can more easily study the structure of elements of $\mathrm{Hom}(H, G)$ and $\mathrm{End}(G)$. Specifically, we will show that the closure of H in G is equivalent to three conditions.

If instead $H \cong D_{2p}$ for p an odd prime, then H has three normal subgroups: 1, H, and the derived subgroup $[H, H]$ of order p. Although the discussion is much more involved, we will arrive at the following result in Section 5:

Theorem. *If $H \cong D_{2p}$ and H is closed in a finite group G, then either $[G, G] = G^\infty$ or $H = G$.*

We also have the following corollary:

Corollary. *If $H \cong D_{2p}$ is closed in a finite solvable group G, then $H = G$.*

The condition in the corollary that G be solvable appears to be necessary, and we expect to be able to prove the following conjecture:

Conjecture. *For each $p > 3$, there exists a closed embedding of $H \cong D_{2p}$ in a finite group G, where G^∞ is a nonabelian finite simple group.*

In Sections 3 and 4, we will develop the lemmas necessary to prove the theorem. We will prove several facts about the relationship between H and G, first under the condition that (1) only the zero map 0_H

extends uniquely, and then under the stronger condition that (2) H is closed in G.

We assume that the reader is familiar with concepts in elementary group theory; see [1] and [4] for an introduction.

2 An Example

We begin with an example from Aschbacher's paper [2], supplying a proof of the assertion made there without proof.

Example. *Let H and G be simple. Then H is closed in G if and only if*

(i) $\mathrm{Aut}(G)$ *is transitive on subgroups of G which are isomorphic to H;*

(ii) $\mathrm{Aut}_{\mathrm{Aut}(G)}(H) = \mathrm{Aut}(H)$*; and*

(iii) $C_{\mathrm{Aut}(G)}(H) = 1$.

Proof. For each $\varphi \in \mathrm{Hom}(H, G)$, we have $\ker\varphi \trianglelefteq H$. Thus, the fact that H is simple implies $\ker\varphi = 1$ or H, and elements of $\mathrm{Hom}(H, G)$ are either injective homomorphisms or 0_H. Similarly, $\mathrm{End}(G) = \mathrm{Aut}(G) \cup \{0_G\}$.

Note that if $H = 1$, then H is closed in G if and only if $G = H = 1$, which is trivially equivalent to the given conditions. Thus, we need only consider the case when $1 < H \leq G$. Note also that if $G \neq 1$, the map 0_H is the only element of $\mathrm{Hom}(H, G)$ that extends to 0_G, and this extension is always unique.

It is easily verified that the given conditions are necessary for H to be closed in G. If (i) is not satisfied, there are some $K \leq G$ and some $\varphi \in \mathrm{Hom}(H, G)$ such that $H\varphi = K$ but φ does not extend to an element of $\mathrm{Aut}(G)$. If (ii) is not satisfied, there is some $\varphi \in \mathrm{Aut}(H) \leq \mathrm{Hom}(H, G)$ that is not the restriction of an element of $\mathrm{Aut}(G)$, so φ cannot extend to an element of $\mathrm{End}(G)$. If (iii) is not satisfied, there exists $1 \neq \alpha \in C_{\mathrm{Aut}(G)}(H)$, and both α and the identity map 1_G extend 1_H.

Next, we prove that the conditions are sufficient. Define $\theta : N_{\mathrm{Aut}(G)}(H) \to \mathrm{Aut}(H)$ by $\alpha\theta = \alpha|_H$ for each $\alpha \in N_{\mathrm{Aut}(G)}(H)$. Then $C_{\mathrm{Aut}(G)}(H) = \ker\theta = 1$, so θ is injective; hence, every element of $\mathrm{Aut}(H)$ in the image of θ extends uniquely to an element of $\mathrm{Aut}(G)$. Furthermore, (ii) implies θ is surjective. Thus, all elements of $\mathrm{Aut}(H)$ extend uniquely to elements of $\mathrm{End}(G)$.

To see that every element $\varphi \in \mathrm{Hom}(H, G)$ different from 0_H extends to at least one element of $\mathrm{End}(G)$, note that by (i), there exists $\psi \in \mathrm{Aut}(G)$ such that $H\varphi = H\psi$, since $H\varphi \cong H$. Then $\varphi\psi^{-1} \in \mathrm{Aut}(H)$ extends to $\rho \in \mathrm{Aut}(G)$, and $\rho\psi \in \mathrm{Aut}(G)$ induces φ on H.

To see that every element of $\mathrm{Hom}(H, G)$ different from 0_H extends to at most one element of $\mathrm{End}(G)$, suppose $\varphi', \varphi'' \in \mathrm{End}(G)$ satisfy $\varphi'|_H = \varphi''|_H$. Then $(\varphi'(\varphi'')^{-1})|_H = 1_H$, so $\varphi'(\varphi'')^{-1} \in C_{\mathrm{Aut}(G)}(H) = 1$ by (iii), implying $\varphi' = \varphi''$. Hence, the three given conditions are indeed equivalent to H being closed in G. □

3 When 0_H Extends Uniquely

We now prove several facts under the more relaxed condition that the zero homomorphism of a subgroup extends uniquely.

Lemma 3.1. *0_H extends uniquely if and only if* $\mathrm{Hom}\big(G/\langle H^G\rangle, G\big) = 0$.

Proof. Suppose $\varphi \in \mathrm{End}(G)$ is an extension of 0_H. (Such a map exists because 0_G extends 0_H.) Since $\langle H^G\rangle$ is the normal closure of H in G, $H \leq \ker\varphi$ implies $\langle H^G\rangle \leq \ker\varphi$. Then φ factors through $G/\langle H^G\rangle$, and if $\pi : G \to G/\langle H^G\rangle$ is the canonical homomorphism, the map φ corresponds to a unique $\psi \in \mathrm{Hom}(G/\langle H^G\rangle, G)$ such that $\pi\psi = \varphi$. Hence, if 0_H extends uniquely to 0_G, we clearly have $\mathrm{Hom}(G/\langle H^G\rangle, G) = 0$, and if $\mathrm{Hom}(G/\langle H^G\rangle, G) = 0$, we have $\varphi = 0_G$. □

For the remainder of this section, assume $0_H \in \mathrm{Hom}(H, G)$ extends uniquely to $0_G \in \mathrm{End}(G)$.

Lemma 3.2. *If G^∞ is the last term in the derived series of G, then $G = \langle H^G\rangle G^\infty$.*

Proof. Recall that the derived series $\{G^i\}$ is defined by the rules that $G^0 = G$ and $G^{i+1} = [G^i, G^i]$ for $i \geq 0$. Furthermore G^∞ is the smallest normal subgroup of G such that G/G^∞ is solvable.

Now suppose $G \neq \langle H^G\rangle G^\infty$, and let the quotient map by G^∞ be denoted by a star. Then $\langle H^G\rangle \trianglelefteq G$ implies $\langle H^G\rangle^* \trianglelefteq G^*$, and since G^* is solvable, so is $G^*/\langle H^G\rangle^*$. Thus, if $X^*/\langle H^G\rangle^*$ is the last term in a composition series of $G^*/\langle H^G\rangle^*$, we have $|G^*/\langle H^G\rangle^* : X^*/\langle H^G\rangle^*| = |G : X| = p$, where p is a prime factor of $|G|$ and $\langle H^G\rangle \leq X \trianglelefteq G$. (The case $G = 1$ trivially implies $G = \langle H^G\rangle G^\infty$.)

By Cauchy's Theorem, there exists $Y \leq G$ with $|Y| = p$. Then G/X and Y are cyclic groups of the same order, so $G/X \cong Y$, and there exists an isomorphism $\varphi : G/X \to Y$. Since $\langle H^G\rangle \leq X$, there exists a homomorphism $\psi : G/\langle H^G\rangle \to G/X$. Then $0 \neq \psi\varphi \in \mathrm{Hom}\big(G/\langle H^G\rangle, G\big)$, which by Lemma 3.1 contradicts the fact that 0_H extends uniquely. □

Remark 3.3. If G is solvable, then $G^\infty = 1$, so Lemma 3.2 states that if 0_H extends uniquely, then $G = \langle H^G\rangle$. The converse is also true in this case, by Lemma 3.1 and the fact that $\mathrm{Hom}\big(G/\langle H^G\rangle, G\big) = \mathrm{Hom}(G/G, G) = 0$.

Lemma 3.4. *If G is nilpotent and $P \leq G = \langle P^G\rangle$, then $G = P$.*

Proof. We prove for every $K \leq G$ that in fact $K \trianglelefteq\trianglelefteq G$. We induct on the index of K in G. If $|G : K| = 1$, then the claim is trivially true. If instead $K < G$, then the fact that G is nilpotent implies $K < N_G(K)$. Since $K \trianglelefteq N_G(K)$ and $|G : N_G(K)| < |G : K|$, we have by induction that $N_G(K) \trianglelefteq\trianglelefteq G$, so $K \trianglelefteq\trianglelefteq G$, as desired. From $P \trianglelefteq\trianglelefteq G$ and $G = \langle P^G\rangle$, we then obtain $G = P$. □

Lemma 3.5. *Let $N \trianglelefteq G$, and let the quotient map by N be denoted with a star. Then $\langle H^G\rangle^* = \big\langle (H^*)^{G^*}\big\rangle$.*

Proof. Since $\langle H^G\rangle \trianglelefteq G$, we have $\langle H^G\rangle^* \trianglelefteq G^*$. Also, $H^* \leq \langle H^G\rangle^*$, so clearly $\big\langle (H^*)^{G^*}\big\rangle \leq \langle H^G\rangle^*$. Since $\langle H^G\rangle = \langle h^g : h \in H, g \in G\rangle$ and $(h^g)^* = (g^{-1}hg)^* = (g^{-1})^*h^*g^* = (h^*)^{g^*}$, we also have $\langle H^G\rangle^* \leq \big\langle (H^*)^{G^*}\big\rangle$, as desired. □

Lemma 3.6. *If L^∞ is the last term in the lower central series of G, then $G = HL^\infty$.*

Proof. Recall that the lower central series $\{L^i\}$ is defined by the rules that $L^0 = G$ and $L^{i+1} = [L^i, G]$ for $i \geq 0$. Furthermore L^∞ is the smallest normal subgroup of G such that G/L^∞ is nilpotent.

Since $G^\infty \leq L^\infty$, Lemma 3.2 implies $G = \langle H^G\rangle G^\infty = \langle H^G\rangle L^\infty$. Let the quotient map by L^∞ be denoted by a star, and suppose $\big\langle (H^*)^{G^*}\big\rangle = K^* \trianglelefteq G^*$, where $L^\infty \leq K \neq G$. Lemma 3.5 shows that $\langle H^G\rangle^* = K^* \trianglelefteq G^*$, so $\langle H^G\rangle \leq K \trianglelefteq G$. Then $\langle H^G\rangle L^\infty \leq KL^\infty = K \neq G$, a contradiction. Hence $\big\langle (H^*)^{G^*}\big\rangle = G^*$.

Since G^* is nilpotent, Lemma 3.4 then implies $G^* = H^*$. Thus, for each $g \in G$, there exists $h \in H$ such that $gL^\infty = hL^\infty$, so $g \in hL^\infty$ and $G \leq HL^\infty$. Clearly $HL^\infty \leq G$, so $G = HL^\infty$, as desired. □

Remark 3.7. If G is nilpotent, then $L^\infty = 1$, so Lemma 3.6 implies $H = G$.

Corollary 3.8. $G = H[G, G]$.

Proof. This follows from Lemma 3.6 and the fact that $L^\infty \leq [G, G]$. □

4 When H is Closed in G

We now prove facts about closed embeddings of groups, which we later apply to the specific case $H \cong D_{2p}$.

Lemma 4.1. *Let $P \leq G$, and let 0_P and 1_P extend uniquely to elements of* $\mathrm{End}(G)$*. Then*
(1) $C_G(P) = Z(G)$, and
(2) if P is abelian, then $G = P$.

Proof. Clearly $Z(G) \leq C_G(P)$. For each $g \in C_G(P)$, we have $c_g|_P = 1_P$, where c_g is the inner automorphism on G induced by g. Since 1_P extends uniquely, we have $c_g = 1_G$, so $g \in Z(G)$ and $C_G(P) \leq Z(G)$, proving (1).

If P is abelian, then $P \leq C_G(P) = Z(G)$. Thus $G = C_G(P) = Z(G)$, so G is abelian and therefore nilpotent. Since 0_P extends uniquely, Remark 3.7 implies $G = P$, as desired. □

For the remainder of this section, assume H is closed in G.

Lemma 4.2. *Given $\varphi \in$* $\mathrm{End}(G)$*, each element of* $\mathrm{Hom}(H\varphi, G\varphi)$ *extends to at most one element of* $\mathrm{End}(G\varphi)$.

Proof. Suppose there exist $\theta, \psi \in \mathrm{End}(G\varphi)$ such that $\theta|_{H\varphi} = \psi|_{H\varphi}$ and $\theta \neq \psi$. Then $\varphi\theta, \varphi\psi \in \mathrm{End}(G)$ and $(\varphi\theta)|_H = (\varphi\psi)|_H \in \mathrm{Hom}(H, G)$, but $\varphi\theta \neq \varphi\psi$, contradicting the fact that H is closed in G. □

Lemma 4.3. *Let $\varphi \in$* $\mathrm{End}(G)$*, and let $H\varphi$ be abelian. Then $G\varphi = H\varphi$.*

Proof. By Lemma 4.2, each element of $\mathrm{Hom}(H\varphi, G\varphi)$ extends to at most one element of $\mathrm{End}(G\varphi)$. In particular, $0_{H\varphi}$ and $1_{H\varphi}$ extend to $0_{G\varphi}$ and $1_{G\varphi}$ respectively, so these extensions are unique. The result then follows from part (2) of Lemma 4.1. □

Lemma 4.4. *Let Δ be a set of cyclic subgroups of G. Let Γ be the set of $K \trianglelefteq H$ such that H/K is isomorphic to an element of Δ. Then $H \cap [G, G] \leq \bigcap_{K\in\Gamma} K$.*

Proof. For each $K \in \Gamma$, there exist $X \in \Delta$ and $\varphi \in \mathrm{Hom}(H, G)$ such that $K = \ker\varphi$ and $X = H\varphi$. Then φ extends to $\varphi_K \in \mathrm{End}(G)$, and $K = H \cap \ker\varphi_K$. Also $H\varphi$ is cyclic and therefore abelian, so $G\varphi_K = H\varphi$ by Lemma 4.3.

Since $G\varphi_K$ is abelian, we have $[G, G] \leq \ker\varphi_K$. Thus $[G, G] \leq \bigcap_{K\in\Gamma} \ker\varphi_K$, and

$$H \cap [G, G] \leq \bigcap_{K\in\Gamma} H \cap \ker\varphi_K = \bigcap_{K\in\Gamma} K,$$

as desired. ☐

Lemma 4.5. *$H \cap [G,G] = [H,H]$.*

Proof. Clearly $[H,H] \leq H \cap [G,G]$. To prove the reverse containment, let the quotient map by $[H,H]$ be denoted with a star. Since H^* is abelian, we may write $H^* = R_1^* \times \cdots \times R_n^*$, where the R_i^*'s are cyclic subgroups of H^*. Let K_i be the preimage of $\prod_{j\neq i} R_j^*$, and let $r_i \in H$ be such that $R_i^* = \langle r_i^* \rangle$.

We first show that $|R_i^*|$ divides $|r_i|$. If $|R_i^*| = k$, then k is the smallest positive integer such that $r_i^k \in [H,H]$. If $|r_i| = l$, then $r_i^l = 1 \in [H,H]$, so $k \nmid l$ provides a contradiction to the minimality of k.

Now let $X_i = \langle r_i^{l/k} \rangle \leq G$. Apply Lemma 4.4 to $\Delta = \{X_1, \ldots, X_n\}$ and note that $H/K_i \cong R_i^* \cong X_i$, so $H \cap [G,G] \leq \bigcap_{K\in\Gamma} K \leq \bigcap K_i = [H,H]$, where the last equality follows from the fact that $\bigcap \prod_{j\neq i} R_j^* = 1$. Hence $H \cap [G,G] = [H,H]$, as desired. ☐

Corollary 4.6. *$G/[G,G] \cong H/[H,H]$.*

Proof. We use Lemma 4.5 and the fact that $G = H[G,G]$, from Corollary 3.8. Since $[G,G] \trianglelefteq G$, the Second Isomorphism Theorem implies $G/[G,G] = H[G,G]/[G,G] \cong H/H\cap[G,G] = H/[H,H]$. ☐

5 When H is Isomorphic to D_{2p}

Throughout Section 5, assume $H \cong D_{2p}$ for p an odd prime, and H is closed in a finite group G.

Lemma 5.1. *Let $L = [G,G]$. Then $|G/L| = 2$.*

Proof. Corollary 4.6 states that $G/L \cong H/[H,H]$. Since $[H,H]$ is the derived subgroup of order p in H, we have $|G/L| = 2$. ☐

Lemma 5.2. *If $\varphi \in \mathrm{Hom}(H,G)$ and $\psi \in \mathrm{End}(G)$ is the unique extension of φ, then exactly one of the following holds:*

(1) $\varphi = 0_H$ and $\psi = 0_G$;
(2) $\ker\varphi = [H,H]$ and $\ker\psi = L$; or
(3) φ is injective.

Proof. Since $\ker\varphi \trianglelefteq H$, the subgroup $\ker\varphi$ is one of H, $[H,H]$, and 1. The first case implies $\varphi = 0_H$, which extends uniquely to 0_G, giving (1).

In the second case, $H\psi = H\varphi \cong H/[H,H] \cong \mathbb{Z}/2$ is abelian, so Lemma 4.3 implies $G/\ker\psi \cong G\psi = H\psi$ is abelian as well. But G/L is the largest abelian quotient of G, so $L \leq \ker\psi$. Lemma 5.1 gives $|G/L| = 2$; since $\ker\psi \neq G$, we must have $\ker\psi = L$, giving (2).

The third case, $\ker\varphi = 1$, is clearly equivalent to (3). ☐

Lemma 5.3. *If $\psi \in \mathrm{End}(G)$ and $G\psi \cong H$, then $G = H$.*

Proof. Let $K = \ker\psi$. Then $G/K \cong G\psi \cong H$, so $|G| = |K||H|$. Since $|G/K| \neq 2$, necessarily $K \neq L$. Also $\psi \neq 0_G$, so Lemma 5.2 implies $\psi|_H$ is injective. Hence $K\cap H = \ker(\psi|_H)\cap H = 1$. Then $KH \leq G$ and $|KH| = |K||H| = |G|$ imply $G = KH$. Since $K \trianglelefteq G$, the group G is the semidirect product of K and H, and the map $\theta : G \to H$ defined by $\theta(kh) = h$ for any $k \in K$ is a homomorphism. But then $\theta|_H = 1_H$, so $\theta = 1_G$ and $kh = \theta(kh) = h$. Hence $K = 1$ and $G = H$. ☐

Lemma 5.4. *Let t be an involution in H. Then*

(1) $G = L\langle t\rangle$, and
(2) $L = L^\infty$.

Proof. Lemma 4.5 gives $H \cap L = [H,H]$, so $L \cap \langle t\rangle = 1$. Since $L \trianglelefteq G$, we have $L\langle t\rangle \leq G$. But $|G|/|L| = 2$ from Lemma 5.1, so $|L\langle t\rangle| = |L||\langle t\rangle| = |G|$ and $G = L\langle t\rangle$, giving (1).

From Lemma 3.6, we have $G = HL^\infty$. Furthermore $L^\infty \leq L$, so $H \cap L^\infty \leq H \cap L$. Since $|H\cap L| = p$, we have $|H\cap L^\infty| = 1$ or p. But $[H,H] = L^\infty(H) \leq L^\infty$ and $[H,H] \leq H$, so $|H \cap L^\infty| \neq 1$. From Corollary 3.8, we also have $G = HL$. Then $|H||L|/|H\cap L| = |G| = |H||L^\infty|/|H\cap L^\infty|$ implies $L^\infty = L$, giving (2). ☐

Lemma 5.5. *Let the quotient map by $[L,L]$ be denoted by a star. Then $L^* = [L^*, t^*]$. In particular, each element of L^* is inverted by t^*.*

Proof. Under the quotient map by $[L,L]$, the results of Lemma 5.4 become $G^* = L^*\langle t^*\rangle$ and $L^* = (L^\infty)^*$. Since $(L^\infty)^* = [(L^\infty)^*, G^*]$, we then have $L^* = [L^*, G^*] = [L^*, L^*\langle t^*\rangle]$. Furthermore L^* is abelian, so $L^* = [L^*, \langle t^*\rangle] = [L^*, t^*]$.

Now $t \notin [L,L]$ is an involution, so t^* is also an involution. If $(l^*)^{-1}t^*l^*t^*$ is a generator of L^*, then $t^*((l^*)^{-1}t^*l^*t^*)t^* = t^*(l^*)^{-1}t^*l^*$, so each generator of L^* is inverted by t^*. If $l_1^*, l_2^* \in L^*$ are both inverted by t^*, then $t^*(l_1^*l_2^*)t^* = (t^*l_1^*t^*)(t^*l_2^*t^*) = (l_1^*)^{-1}(l_2^*)^{-1} = (l_2^*)^{-1}(l_1^*)^{-1} = (l_1^*l_2^*)^{-1}$, so $l_1^*l_2^*$ is also inverted by t^*. Hence, all elements of L^* are inverted by t^*. ☐

Lemma 5.6. *All elements of L^* have odd order.*

Proof. Let $L_0^* \leq L^*$ be the subgroup of elements of odd order, and let S^* be a Sylow 2-group of L^*. Then

$L^* = L_0^* S^*$, so $G^* = L_0^* S^* \langle t^* \rangle$. Since $S^* \langle t^* \rangle$ is a 2-group (and hence nilpotent), we have $L_0^* \cap S^* \langle t^* \rangle = 1$. Also $L_0^* \trianglelefteq G^*$, so $G^*/L_0^* \cong S^* \langle t^* \rangle$. Then G^*/L_0^* is nilpotent and $L^* = L^{\infty *} \leq L_0^*$, so $L_0^* = L^*$, as desired. ☐

Lemma 5.7. *Either* $[G, G] = G^\infty$ *or* $H = G$.

Proof. Note that $L^* = 1$ implies $L = [L, L]$, which is equivalent to $[G, G] = G^\infty$. Lemma 5.5 shows that t^* inverts L^*, so t^* acts on each subgroup of L^*. Thus, if $L^* \neq 1$, there exists a maximal subgroup $M^* < L^*$ with $M^* \trianglelefteq G^*$. Let M be the preimage of M^* in G, and let the quotient map by M be denoted by a bar. By Lemma 5.6 and the fact that every maximal subgroup of an abelian group has prime index, we have $|L : M| = |L^* : M^*| = q$ for some odd prime q, and $|\bar{G}| = 2q$.

If we now pick $l \in L \backslash M$, we have $M \trianglelefteq L$ and $\langle l \rangle \leq L$, so $M\langle l \rangle \leq L$. Since M is a maximal subgroup of L, we also have $M\langle l \rangle = L$. Lemma 5.4 gives $G = L\langle t \rangle = \langle L, t \rangle$, which implies $\bar{G} = \langle \bar{l}, \bar{t} \rangle$. Since t^* is an involution and l^* is inverted by t^* (by Lemma 5.5), also $\bar{t}$ is an involution and $\bar{l}$ is inverted by $\bar{t}$. Thus $\bar{G} \cong D_{2q}$.

Now $|\bar{G}|$ is odd, so $\bar{t}^{\bar{l}}$ is an involution in $\bar{G}$. Hence $\bar{G} = \langle \bar{t}^{\bar{l}}, \bar{t} \rangle$ and $|\langle \bar{t}^{\bar{l}} \bar{t} \rangle| = q$. Then q divides $|t^l t|$, and there exists $y \in \langle t^l t \rangle$ such that $|y| = q$. Furthermore $t(t^l t)t = t(l^{-1} t l t)t = (t t^l)^{-1}$, so all elements of $\langle t^l t \rangle$, including y, are inverted by t. Hence $E = \langle t, y \rangle \cong D_{2q} \cong \bar{G}$, and there exists a surjective homomorphism $\psi : G \to E$, factoring through G/M, with $\ker \psi = M$.

We now apply Lemma 5.2 to $\psi \in \mathrm{End}(G)$ and $\varphi = \psi|_H$. Clearly $\psi \neq 0_G$ and $\ker \psi = M \neq L$, so φ is an injection. Then $H\varphi \leq E$ implies $2p | 2q$, so $p = q$ and $E \cong H$. Combining ψ with this isomorphism provides $\theta \in \mathrm{End}(G)$ with $G\theta \cong H$, so $G = H$ by Lemma 5.3, completing the proof. ☐

Corollary 5.8. *If* G *is solvable, then* $H = G$.

Proof. Since $D_{2p} \leq G$, the group G is not abelian. Hence $G^\infty = 1 \neq [G, G]$, and $H = G$ by Lemma 5.7. ☐

6 Conclusion

We have discussed general closed embeddings of finite groups and studied the specific cases when both H and G are simple and when $H \cong D_{2p}$. We have also proved several facts about embeddings of finite groups under the weaker condition that 0_H extends uniquely. In the future, we hope to apply similar methods to study closed embeddings of other classes of finite groups.

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