

MASSACHUSETTS INSTITUTE OF TECHNOLOGY  
DEPARTMENT OF PHYSICS  
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LECTURE 4:  
WRAP UP OF VECTOR CALCULUS:  
POISSON & LAPLACE EQUATIONS; CURL

## 4.1 Summary: Vector calculus so far

We have learned several mathematical operations which fall into the category of vector calculus. In Cartesian coordinates, these operations can be written in very compact form using the following operator:

$$\vec{\nabla} \equiv \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} .$$

The first vector calculus operation we learned is the *gradient*. It converts the electric *potential* into the electric *field*:

$$\begin{aligned} \vec{E} &= -\mathbf{grad} \phi \\ &= -\vec{\nabla} \phi . \end{aligned}$$

The second operation is the *divergence*, which relates the electric field to the charge density:

$$\mathbf{div} \vec{E} = 4\pi\rho .$$

Via *Gauss's theorem* (also known as the *divergence theorem*), we can relate the flux of any vector field  $\vec{F}$  through a closed surface  $S$  to the integral of the divergence of  $\vec{F}$  over the volume enclosed by  $S$ :

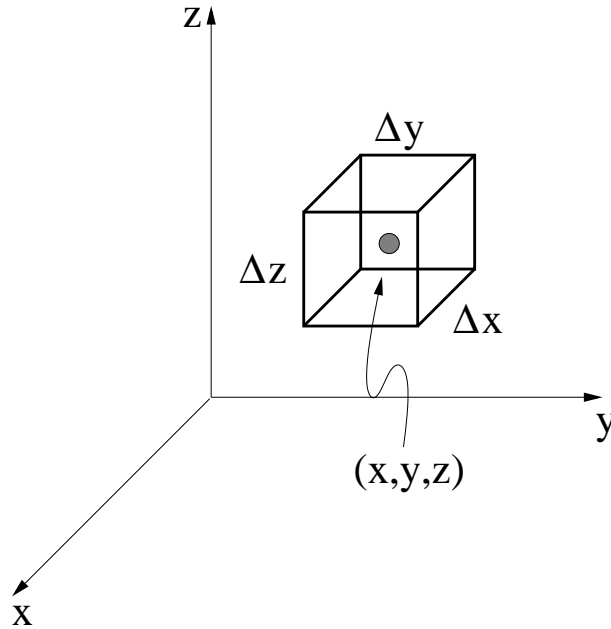
$$\oint_S \vec{F} \cdot d\vec{A} = \int_V \mathbf{div} \vec{F} dV .$$

This follows from the definition of the divergence:

$$\mathbf{div} \vec{F} = \lim_{V \rightarrow 0} \frac{\oint_S \vec{F} \cdot d\vec{A}}{V} .$$

## 4.2 Divergence in Cartesian coordinates

So far, we've only defined the divergence as a particular limit. We now want to develop a concrete calculation showing its value. To do so, consider an infinitesimal cube with sides  $\Delta x$ ,  $\Delta y$ , and  $\Delta z$ , centered on the coordinate  $x$ ,  $y$ ,  $z$ :



[The large dot in the center of this cube is only there to mark the coordinate  $(x, y, z)$  — it doesn't mean anything else.] We take this region to be filled with a vector field  $\vec{F}$ . To begin, we want to compute the flux of  $\vec{F}$  over this cube. Let's look at the flux over just the  $z$  faces (the top and bottom):

$$\begin{aligned} \Delta\Phi_z &= \int_{\text{top \& bottom}} \vec{F} \cdot d\vec{A} \\ &\simeq \Delta x \Delta y [F_z(x, y, z + \Delta z/2) - F_z(x, y, z - \Delta z/2)] . \end{aligned}$$

Since the cube is taken to be very small (indeed, we will soon take the limit  $\Delta x, \Delta y, \Delta z \rightarrow 0$ ), we approximate the integral. Notice that the contribution from the bottom face ( $z - \Delta z/2$ ) enters with a minus sign, consistent with that face pointing down.

Now, to evaluate the function under the integrand, we make a Taylor expansion about  $z$ :

$$\begin{aligned} \Delta\Phi_z &\simeq \Delta x \Delta y \left[ F_z(x, y, z) + \frac{\Delta z}{2} \frac{\partial F_z}{\partial z} \right] - \Delta x \Delta y \left[ F_z(x, y, z) - \frac{\Delta z}{2} \frac{\partial F_z}{\partial z} \right] \\ &\simeq \Delta x \Delta y \Delta z \frac{\partial F_z}{\partial z} . \end{aligned}$$

Repeating this exercise to find the fluxes through the other four sides, we find

$$\begin{aligned} \Delta\Phi_x &\simeq \Delta x \Delta y \Delta z \frac{\partial F_x}{\partial x} \\ \Delta\Phi_y &\simeq \Delta x \Delta y \Delta z \frac{\partial F_y}{\partial y} . \end{aligned}$$

We now have enough information to evaluate the divergence of  $\vec{F}$ :

$$\begin{aligned}
 \operatorname{div} \vec{F} &= \lim_{V \rightarrow 0} \frac{\int \vec{F} \cdot d\vec{A}}{V} \\
 &= \lim_{\Delta x, \Delta y, \Delta z \rightarrow 0} \frac{\Delta x \Delta y \Delta z (\partial F_x / \partial x + \partial F_y / \partial y + \partial F_z / \partial z)}{\Delta x \Delta y \Delta z} \\
 &= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \\
 &\equiv \vec{\nabla} \cdot \vec{F}.
 \end{aligned}$$

On the last line, we've taken advantage of the fact that we can think of the gradient operator as a vector. In Cartesian coordinates, the divergence is nothing more than combination of derivatives one finds by taking the dot product of this operator with a vector field. It should be *strongly* emphasized at this point, however, that this only works in Cartesian coordinates. In spherical coordinates or cylindrical coordinates, the divergence is not just given by a dot product like this!

#### 4.2.1 Example: Recovering $\rho$ from the field

In Lecture 2, we worked out the electric field associated with a sphere of radius  $a$  containing uniform charge density  $\rho$ :

$$\begin{aligned}
 \vec{E}(r) &= \frac{4}{3}\pi\rho r\hat{r} & r \leq a \\
 &= \frac{4\pi\rho a^3/3}{r^2}\hat{r} & r > a.
 \end{aligned}$$

Recall that we found this by applying the integral form of Gauss's law. Breaking it up into Cartesian components, we have for  $r \leq a$

$$\begin{aligned}
 E_x &= \frac{4}{3}\pi\rho x \\
 E_y &= \frac{4}{3}\pi\rho y \\
 E_z &= \frac{4}{3}\pi\rho z;
 \end{aligned}$$

for  $r > a$ ,

$$\begin{aligned}
 E_x &= \frac{4\pi\rho a^3 x/3}{(x^2 + y^2 + z^2)^{3/2}} \\
 E_y &= \frac{4\pi\rho a^3 y/3}{(x^2 + y^2 + z^2)^{3/2}} \\
 E_z &= \frac{4\pi\rho a^3 z/3}{(x^2 + y^2 + z^2)^{3/2}}.
 \end{aligned}$$

(We used the facts that  $r\hat{r} = x\hat{x} + y\hat{y} + z\hat{z}$ , and  $\hat{r}/r^2 = r\hat{r}/r^3$  in here.)

OK, let's now evaluate  $\vec{\nabla} \cdot \vec{E}$ . For  $r \leq a$ , we have

$$\frac{\partial E_x}{\partial x} = \frac{\partial E_y}{\partial y} = \frac{\partial E_z}{\partial z} = \frac{4}{3}\pi\rho,$$

so

$$\vec{\nabla} \cdot \vec{E} = \frac{4}{3}\pi\rho + \frac{4}{3}\pi\rho + \frac{4}{3}\pi\rho = 4\pi\rho$$

exactly as advertised! The region  $r > a$  is a bit messier:

$$\frac{\partial E_x}{\partial x} = \frac{\partial}{\partial x} \frac{4\pi\rho a^3 x/3}{(x^2 + y^2 + z^2)^{3/2}} \quad (1)$$

$$= \frac{4\pi\rho a^3}{3} \left[ \frac{1}{(x^2 + y^2 + z^2)^{3/2}} - \frac{3x^2}{(x^2 + y^2 + z^2)^{5/2}} \right]. \quad (2)$$

Repeating for  $y$  and  $z$  components and adding, we have

$$\vec{\nabla} \cdot \vec{E} = \frac{4\pi\rho a^3}{3} \left[ \frac{3}{(x^2 + y^2 + z^2)^{3/2}} - \frac{3x^2 + 3y^2 + 3z^2}{(x^2 + y^2 + z^2)^{5/2}} \right] \quad (3)$$

$$= \frac{4\pi\rho a^3}{3} \left[ \frac{3}{(x^2 + y^2 + z^2)^{3/2}} - \frac{3}{(x^2 + y^2 + z^2)^{3/2}} \right] \quad (4)$$

$$= 0. \quad (5)$$

Since there is no charge in the exterior region —  $\rho = 0$  for  $r > a$  — this is exactly what  $\vec{\nabla} \cdot \vec{E}$  *should* give.

### 4.3 Laplacian operator and Poisson's equation

The div and grad operations can be combined to make an equation that relates the potential  $\phi$  to the charge density  $\rho$ :

$$\mathbf{div\ grad\ } \phi = -4\pi\rho.$$

In Cartesian coordinates, this takes a very nice, simple form:

$$\vec{\nabla} \cdot (\vec{\nabla}\phi) = -4\pi\rho$$

or

$$\nabla^2\phi = -4\pi\rho.$$

The operator  $\nabla^2$  is, in Cartesian coordinates, exactly what'd you expect taking  $\vec{\nabla}$  and dotting it into itself:

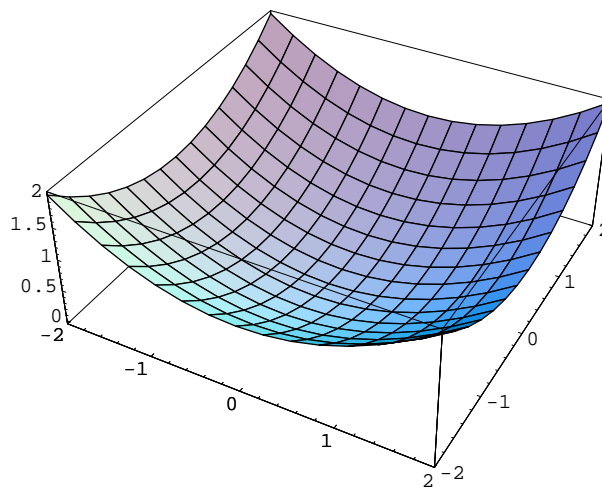
$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

The equation  $\nabla^2\phi = -4\pi\rho$  is called Poisson's equation; the operator  $\nabla^2$  is called the “Laplacian”.

For a two dimensional function  $\phi(x, y)$ , the Laplacian tells us about the *average* curvature of the function. Let's look at a couple of examples. First, take a potential that is a bowl-like paraboloid in the  $x - y$  plane:

$$\phi(x, y) = \frac{a}{4} (x^2 + y^2) ,$$

which has the appearance



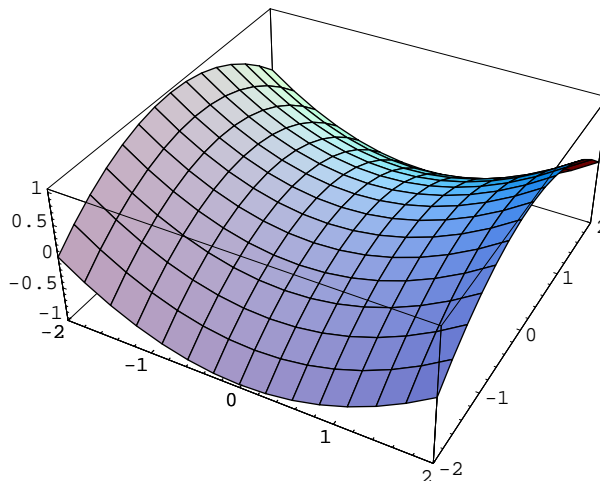
For this function,  $\nabla^2\phi = \partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 = a$  — the function has positive curvature everywhere, reflected by the fact that the bowl-like function curls up everywhere. There is a minimum at  $x = 0, y = 0$ .

## 4.4 Laplace's equation

Second, look at a potential that has a hyperboloidal shape in the plane:

$$\phi(x, y) = \frac{a}{4} (x^2 - y^2) .$$

This function has the appearance



In this case, we find  $\nabla^2\phi = \partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 = 0$ . The average curvature of this function is zero — it curves up in  $x$ , and down in  $y$ . There is *no* minimum (or maximum) in this case, only a saddle point at  $x = 0, y = 0$ .

The case  $\nabla^2\phi = 0$  — that is, the charge density  $\rho = 0$  — is a special case of Poisson's equation called *Laplace's equation*. Laplace didn't just attach his name to Poisson's equation in this special case out of some kind of greedy spite. The two equations really merit separate treatment and different names, since their solutions (and methods for solving them) have rather different properties. (Indeed, Laplace was Poisson's teacher; the Laplace equation predates Poisson's by several decades.)

## 4.5 Earnshaw's Theorem

A potential that satisfies Laplace's equation has a very important property: if  $\nabla^2\phi = 0$  in some region, then  $\phi$  has no local minimum (or maximum) in that region. For there to be a minimum, a function must have positive second derivative in every possible direction. This just isn't possible if

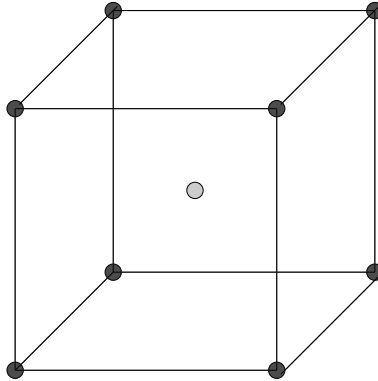
$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0 .$$

Suppose we manage to locate a place where  $\phi$  has a minimum in the  $x$  direction and a minimum in the  $y$  direction. We know, then, that  $\partial^2\phi/\partial x^2 > 0$  and  $\partial^2\phi/\partial y^2 > 0$ . The only way for  $\phi$  to satisfy Laplace's equation is if  $\partial^2\phi/\partial z^2 < 0$ . This location is a minimum in  $x$  and  $y$ , but it is *not* a minimum in  $z$ . The best we can do is find a saddle point, a location that is a minimum along two directions, but a maximum along the third.

The non-existence of minima for regions in which  $\rho = 0$  has an important physical consequence: it means that it is impossible to hold a charge in stable equilibrium with

electrostatic fields. Any object that is in stable equilibrium must be at a minimum of its potential energy. Since the potential energy of a charge in an electrostatic field is just  $U = q\phi$ , it follows that stable equilibrium requires a minimum in the electric potential. Such a minimum just isn't possible when  $\rho = 0$  — we *must* have  $\nabla^2\phi = 0$ . This result goes under the name *Earnshaw's Theorem*.

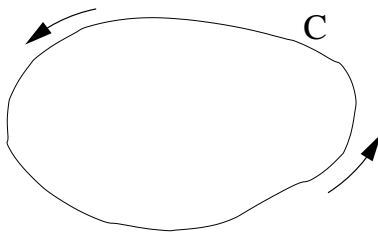
This means that any configuration made out of pure point charges cannot be stable. For example, if we glue positive charges to the corners of a cube, it is impossible to stably hold any kind of charge inside that cube:



All the forces cancel out on a charge at the exact center — but, if we displace it just slightly away from the center, it will rapidly fall out of equilibrium and come shooting out of the cube. This result was very disturbing to scientists in the late 19th and early 20th centuries, since it implied that atoms could not be stable — the electron should fall into the proton after a few microseconds. It wasn't until we developed an understanding of quantum mechanics that the stability of matter made any sense.

## 4.6 The curl

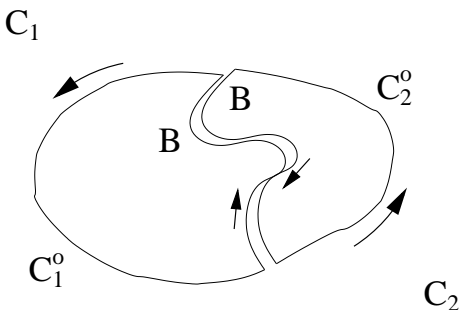
We wrap up this lecture with the final instrument in our vector calculus arsenal. We begin by thinking about the line integral of a vector field  $\vec{F}$  around some closed loop  $C$ :



This quantity is called the *circulation*  $\Gamma$  of  $\vec{F}$ :

$$\Gamma = \oint_C \vec{F} \cdot d\vec{s}.$$

We now bridge the contour  $C$  with a new path  $B$ . Doing so we make two new loops  $C_1$  and  $C_2$ .  $C_1$  is built from an open loop  $C_1^o$  and the path  $B$  going upwards;  $C_2$  is built from an open loop  $C_2^o$  and the path  $B$  going downwards:



Now, we manipulate the integral in a manner similar to what we did when we derived the divergence:

$$\begin{aligned}
 \Gamma &= \int_{C_1^o} \vec{F} \cdot d\vec{s} + \int_{C_2^o} \vec{F} \cdot d\vec{s} \\
 &= \int_{C_1^o} \vec{F} \cdot d\vec{s} + \int_{C_2^o} \vec{F} \cdot d\vec{s} + \int_B \vec{F} \cdot d\vec{s} + \int_B \vec{F} \cdot (-d\vec{s}) \\
 &= \left( \int_{C_1^o} \vec{F} \cdot d\vec{s} + \int_B \vec{F} \cdot d\vec{s} \right) + \left( \int_{C_2^o} \vec{F} \cdot d\vec{s} - \int_B \vec{F} \cdot d\vec{s} \right) \\
 &= \oint_{C_1} \vec{F} \cdot d\vec{s} + \oint_{C_2} \vec{F} \cdot d\vec{s} \\
 &\equiv \Gamma_1 + \Gamma_2 .
 \end{aligned}$$

The key bit to this analysis is that when we cut a big contour in half, we end up with a path going in one direction on the new “bridging” contour, and in the opposite direction on the other side. Hence, everytime we do this, we get “internal” contours that cancel out, and what’s left is equivalent to the original big contour.

We now repeatedly break up the contour into lots of little bits:

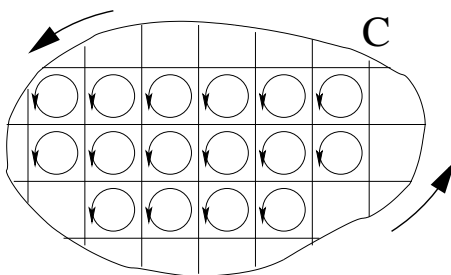
$$\oint_C \vec{F} \cdot d\vec{s} = \sum_{i=1}^{\text{big}} \oint_{C_i} \vec{F} \cdot d\vec{s} ,$$

or

$$\Gamma = \sum_{i=1}^{\text{big}} \Gamma_i .$$



As we keep breaking the object up, we end up covering the *area* enclosed by the contour with a grid of little circulation cells:



and so the total circulation must be proportional to the area that the contour encloses. We thus define a vector called the *curl* of  $\vec{F}$ :

$$\mathbf{curl} \vec{F} \cdot \hat{n} = \lim_{A \rightarrow 0} \frac{\oint_C \vec{F} \cdot d\vec{s}}{A}.$$

The area  $A$  is bounded by the contour  $C$ . The curl is thus circulation of  $\vec{F}$  per unit  $A$ , in the limit in which  $A$  goes to zero. Notice that the curl is a *vector*: it points in the direction  $\hat{n}$  which is normal to the surface. We define this direction with the right hand rule: curl your right-hand fingers in the direction of  $C$ 's circulation, and your thumb defines  $\hat{n}$ .

As with the divergence, an important theorem is connected to the curl. We start with its definition, and then do the breaking up into little bits:

$$\begin{aligned} \Gamma = \oint_C \vec{F} \cdot d\vec{s} &= \sum_{i=1}^{\text{big}} \oint_{C_i} \vec{F} \cdot d\vec{s} \\ &= \sum_{i=1}^{\text{big}} A_i \frac{\oint_{C_i} \vec{F} \cdot d\vec{s}}{A_i} \\ &= \sum_{i=1}^{\text{big}} A_i \mathbf{curl} \vec{F} \cdot \hat{n} \\ &= \sum_{i=1}^{\text{big}} \mathbf{curl} \vec{F} \cdot (A_i \hat{n}) \\ &= \oint_A \mathbf{curl} \vec{F} \cdot d\vec{A}. \end{aligned}$$

In words, the circulation of a vector field  $\vec{F}$  over a closed contour  $C$  equals the flux of the curl of  $\vec{F}$  through the surface bounded by  $C$ . This rule is known as *Stoke's theorem*.

Suppose our vector field  $\vec{F}$  is just an electrostatic field,  $\vec{E}$ . We already know that

$$\oint_C \vec{E} \cdot d\vec{s} = 0$$

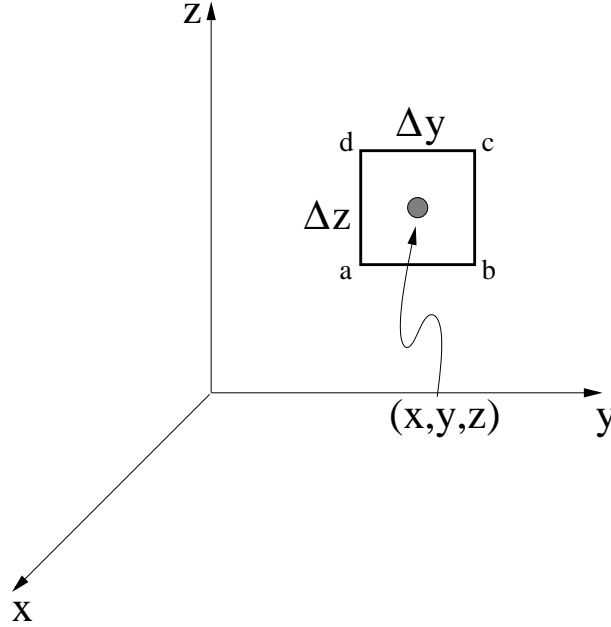
for any contour  $C$  since the electric force is conservative. It follows from this that

$$\boxed{\mathbf{curl} \vec{E} = 0}$$

This is an important point, which is worth emphasizing: a “good” electrostatic field has *no curl*.

## 4.7 The curl in Cartesian coordinates

Consider a square lying in a plane parallel to the  $y-z$  plane, with sides  $\Delta y$  and  $\Delta z$ , centered on the coordinate  $(x, y, z)$ . This square is immersed in a vector field  $\vec{F}$ . Using this square, let's work out what the curl must be.



We approximate the integral along the 4 legs as follows:

$$\begin{aligned} \int_a^b \vec{F} \cdot d\vec{s} &= F_y(x, y, z - \Delta z/2) \Delta y \simeq \left[ F_y(x, y, z) - \frac{\Delta z}{2} \frac{\partial F_y}{\partial z} \right] \Delta y \\ \int_b^c \vec{F} \cdot d\vec{s} &= F_z(x, y + \Delta y/2, z) \Delta z \simeq \left[ F_z(x, y, z) + \frac{\Delta y}{2} \frac{\partial F_z}{\partial y} \right] \Delta z \\ \int_c^d \vec{F} \cdot d\vec{s} &= F_y(x, y, z + \Delta z/2) (-\Delta y) \simeq - \left[ F_y(x, y, z) + \frac{\Delta z}{2} \frac{\partial F_y}{\partial z} \right] \Delta y \\ \int_d^a \vec{F} \cdot d\vec{s} &= F_z(x, y - \Delta y/2, z) (-\Delta z) \simeq - \left[ F_z(x, y, z) - \frac{\Delta y}{2} \frac{\partial F_z}{\partial y} \right] \Delta z . \end{aligned}$$

Adding these together, we have

$$\oint_{abcd} \vec{F} \cdot d\vec{s} = \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \Delta y \Delta z ,$$

which tells us that

$$\begin{aligned} [\mathbf{curl} \vec{F}]_x &= \lim_{\Delta z, \Delta y \rightarrow 0} \frac{\oint_{abcd} \vec{F} \cdot d\vec{s}}{\Delta z \Delta y} \\ &= \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) . \end{aligned}$$

This is the  $x$  component of the curl because, by right-hand rule, that is the orientation of the square.

If we repeat this exercise for squares oriented parallel to the  $x - y$  planes and the  $x - z$  planes, we find

$$\begin{aligned} [\mathbf{curl} \vec{F}]_y &= \left( \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \\ [\mathbf{curl} \vec{F}]_z &= \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right). \end{aligned}$$

There's a nice trick that we can use to work out these two components: we use *cyclic permutation* to flip around the axes. This means we just cyclic the labels, which is equivalent to rotating the axes around. To get  $[\mathbf{curl} \vec{F}]_y$ , take the answer for  $[\mathbf{curl} \vec{F}]_x$ , and permute the coordinate indices:  $x$  becomes  $y$ ,  $y$  becomes  $z$ ,  $z$  becomes  $x$ . Do it one more time, and you've got the answer for  $[\mathbf{curl} \vec{F}]_z$ .

Putting all of this together, we have

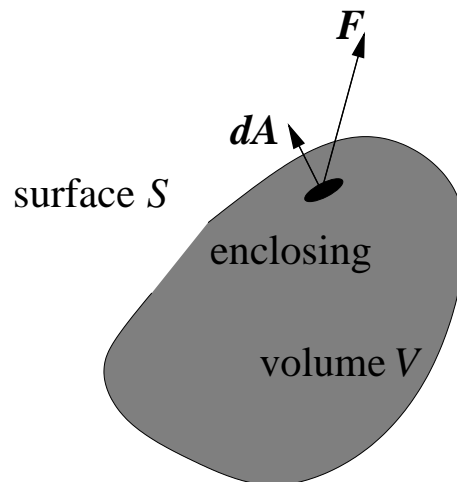
$$\begin{aligned} \mathbf{curl} \vec{F} &= \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \hat{x} + \left( \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \hat{y} + \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{z} \\ &= \vec{\nabla} \times \vec{F}. \end{aligned}$$

Just as the divergence is nicely expressed using the gradient operator and the dot product, the curl is nicely expressed using the gradient operator and the cross product.

## 4.8 Wrap up and summary

That was a lot of math. Let's summarize all of these results:

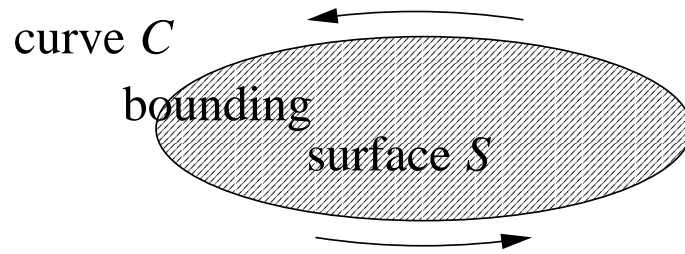
### 4.8.1 Gauss's theorem



$$\oint_S \vec{F} \cdot d\vec{A} = \int_V \mathbf{div} \vec{F} dV$$

The surface  $S$  is the boundary of the volume  $V$ .

### 4.8.2 Stokes's law



$$\oint_C \vec{F} \cdot d\vec{s} = \int_S \mathbf{curl} \vec{F} \cdot d\vec{A}.$$

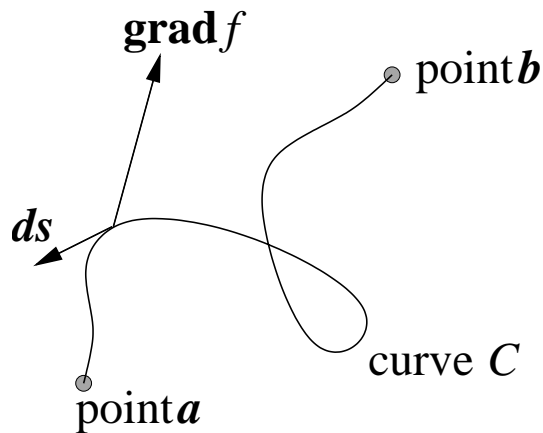
The curve  $C$  is the boundary of the surface  $S$ .

### 4.8.3 Gradient theorem

We only discussed this in detail for the electric potential, but the following rule holds for *any*<sup>1</sup> scalar function:

$$f(\vec{b}) - f(\vec{a}) = \int_{\vec{a}}^{\vec{b}} \vec{\nabla} f \cdot d\vec{s}.$$

This holds for any path from the point  $\vec{a}$  to  $\vec{b}$ :



We will use these results *many* times as this course progresses.

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<sup>1</sup>Except for some really bizarre ones which we will never encounter in this class.