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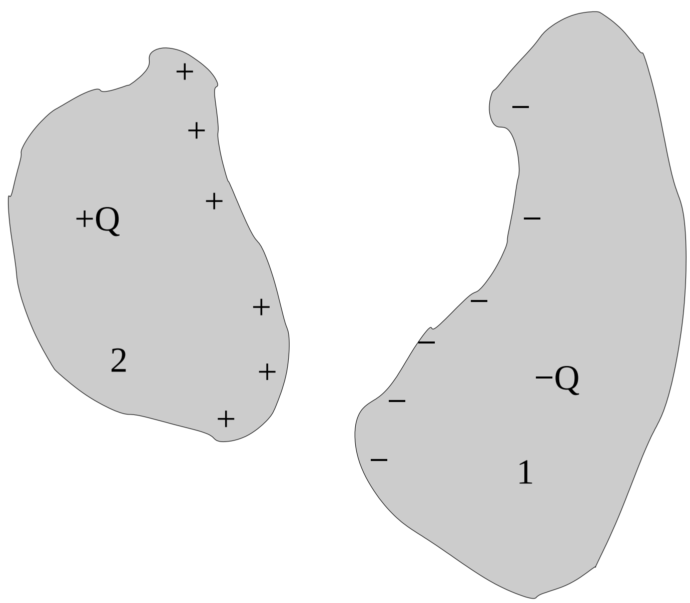
Lecture 6:
Capacitance

6.1 Capacitance

Suppose that I have two chunks of metal. A charge $+Q$ is on one of these chunks, $-Q$ is on the other (so that the system is neutral overall). Each chunk will be at some constant potential. What is the potential difference between the two chunks of metal?



Whatever the potential difference $V \equiv \phi_2 - \phi_1 = -\int_1^2 \vec{E} \cdot d\vec{s}$ turns out to be, it must of course turn out to be independent of the integration path. In fact, as we can show with a little thought, the potential difference V must be proportional to the geometry. This means that the potential difference (or "voltage") between the two chunks must take the form

$$V = (\text{Horribly messy constant depending on geometry}) \times Q \ .$$

The horrible mess that appears in this proportionality law depends only on geometry. In other words, it will depend on the size and shape of the two metal chunks, their relative orientation, and their separation. It does *not* depend on their charge.

This constant is defined as $1/C$, where C is the *Capacitance* of this system. A system like this is then called a *capacitor*. The 1/ may seem a bit wierd; the point is that one usually writes the capacitance formula in a different way: we put

$$Q = CV \ .$$

A key thing to bear in mind when using this formula is that Q refers to the charge *separation* of the system. In other words, when I say that a capacitor is charged up to some level Q,

Figure 1: $Q = CV$.

I mean that I have put $+Q$ on one part of the capacitor, $-Q$ on the other. *The capacitor as a whole remains neutral!!* I emphasize this now because I often find students are somewhat confused by this point, particularly when we start thinking about circuits that have capacitors in them.

6.1.1 Why are the charge and voltage proportional?

The real concern here is, how do we know that when we say double the charge on the capacitor, the charge doesn't move around and cause the dependence to be more complicated? The simple answer is the principle of superposition: whenever we double the amount of charge, the fields that these charges create simply double. Hence, the potential difference (which is just the line integral of the field) must also double.

There's an implied assumption here, though: we are assuming that there is enough charge smeared out on the "plates" of the capacitor to uniformly cover them. Imagine we just had one electron on the minus plate, and one electron "hole" on the plus plate. If we just doubled the charge, the electron on the - plate indeed might move somewhere else. This situation could be messy!

Fortunately, in every situation that is of practical interest, the assumption that the plates are evenly coated with a charge excess is a good one. This is because the elementary charge is so small — even if there's just a tiny tiny amount of charge — say 10^{-12} Coulombs — that means we've got something like 6 million excess electrons.

6.1.2 Units of capacitance

In SI units, we measure charge in Coulombs and potential in Volts, so the unit of capacitance is the Coulomb/Volt. This combination is given a special name: the Farad, or F.

In cgs units, we measure charge in esu and potential in esu/cm. The unit of capacitance is thus just the centimeter! This is actually kind of cool: objects that are big (have lots of centimeters) tend to have a big capacitance.

It's important to know how to convert between these two units: when we are computing capacitance from the system's characteristics, cgs units are nice, since it's just a matter of getting the geometry right. But SI units are used almost exclusively in circuits.

$$\begin{aligned} 1\,\mathrm{cm} &= 1.11 \times 10^{-12}\,\mathrm{F} \\ &\simeq 1\,\mathrm{pF}\ . \end{aligned}$$

"pF" means "picofarad" — one trillionth of a Farad. Capacitors typically come in microfarads or picofarads; this is because (as we shall see in a moment) a Farad turns out to be an enormous amount of capacitance.

6.2 Some examples

6.2.1 Isolated sphere

We take a sphere of radius R and put a charge Q on it. Its potential (relative to infinity) is $V = Q/R$. The sphere's capacitance is therefore

$$C_{\mathrm{sphere}} = R\ .$$

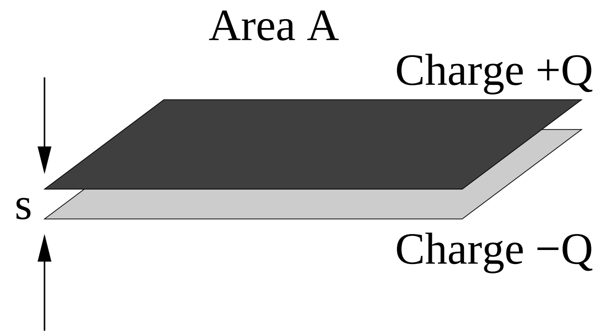
When we began discussing the notion of capacitors, we talked about the potential difference between two conductors. In this case, the second conductor is a "virtual" plate at infinity.

There's one spherical capacitor that we use all the time: the earth. It has a radius (and hence a capacitance) of 6.4×10^8 cm. This is so large that we can effectively take charge from or dump charge onto the earth without changing its electrical potential at all. Converting, the earth has a capacitance $C_{\mathrm{earth}} = 0.0007$ Farad — enormous, but still significantly smaller than a Farad!

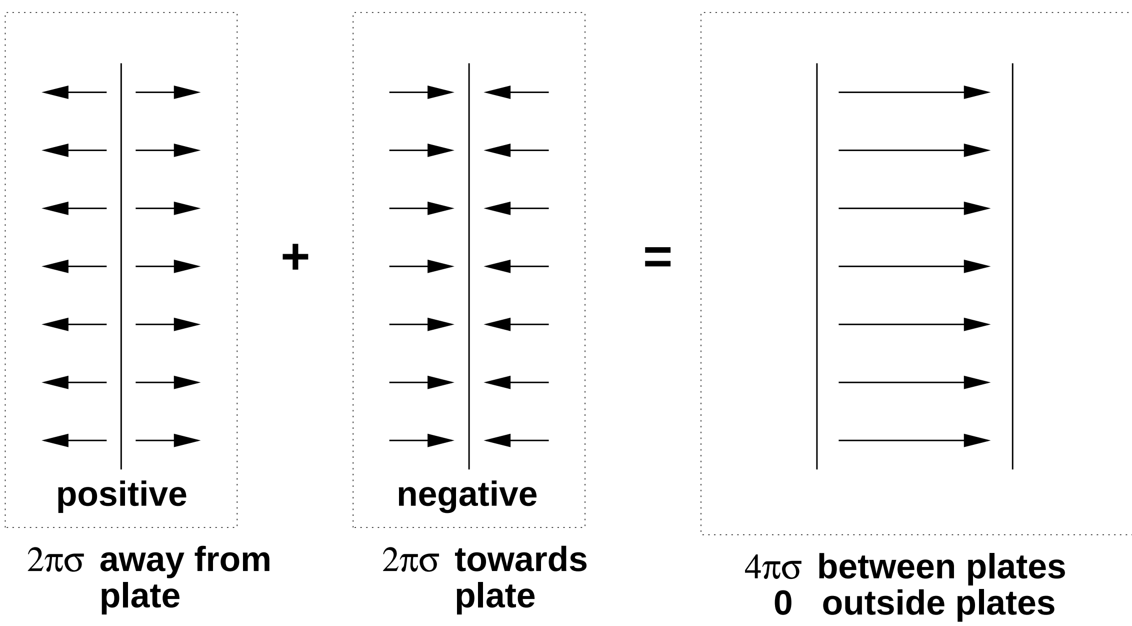
The fact that we can dump charge onto or take charge from the earth without changing its potential appreciably means that it is often convenient to define the earth as the reference point for computing potential. Many circuits take advantage of this by hooking directly to "ground" — a lead that runs into the earth. That "grounded lead" is then defined, by convention, as potential zero. (Students educated in the remnants of the British Empire may have learned to call a circuit that is grounded "earthed".)

6.2.2 Parallel plates

Consider a pair of plates, each with area A and separated by a distance s.



If the plates are close together ($s \ll \sqrt{A}$), we can approximate the field between these plates as those coming from a pair of infinite planes:



Summing an infinite plane with charge density $\sigma = +Q/A$ and $\sigma = -Q/A$ we find

$$E = 4\pi\sigma = 4\pi Q/A$$

between the plates, and zero outside of the plates. The field points from the positive plate to the negative plate, so the potential difference from the negative to the positive plate is

$$V = -\int_{\text{neg}}^{\text{pos}} \vec{E} \cdot d\vec{s} = 4\pi\sigma \int_0^s ds = \frac{4\pi Q s}{A} \; .$$

Rewriting this in the form $Q = CV$, we read off the capacitance:

$$C_{\text{plates}} = \frac{A}{4\pi s} \; .$$

This is an area divided by a length, so it has the correct units.

It's pretty important to know about capacitance in SI units, since most circuit elements are discussed using things like Volts and Farads rather than statvolts and centimeters. Converting is easy: we just need to remember that the electric field in general has a factor of k (the constant from Coulomb's law) attached to it. Hence, the voltage V likewise picks up this factor; rearranging to solve for the capacitance, we find

$$C_{\rm plates} = \frac{1}{k}\frac{A}{4\pi s} \; .$$

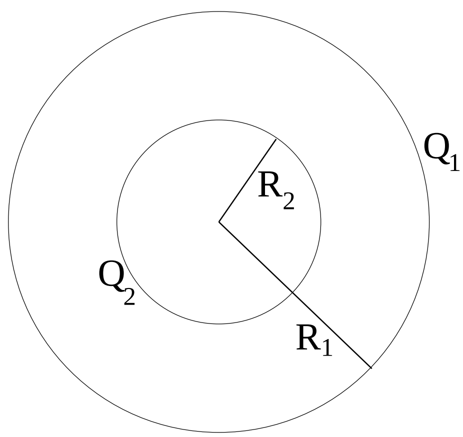
In cgs units, $k = 1$, so this rather trivially reduces to what we had before. In SI units, $k = 1/4\pi\epsilon_0$, and we find

$$C_{\rm plates} = \frac{\epsilon_0 A}{s} \; .$$

Whenever you work out a formula for capacitance in cgs units, you can easily convert to SI by multiplying by $1/k = 4\pi\epsilon_0$.

6.2.3 Nested shells

Return to the example discussed last lecture: a pair of nested spherical shells,



The potential difference we found in this case was

$$\Delta\phi = V = \frac{Q_2}{R_2} - \frac{Q_2}{R_1} \; .$$

Set $Q_2 = Q = -Q_1$:

$$\Delta\phi = V = Q\left(\frac{1}{R_2} - \frac{1}{R_1}\right) \; .$$

Rearranging into the form $Q = CV$, we find

$$C = \frac{R_1 R_2}{R_1 - R_2} \; .$$

Note that if R_1 is just barely larger than R_2, this gives the same result as the parallel plate formula: putting $R_1 = R_2 + s$, we find

$$\begin{aligned} C &= \frac{R_2^2 + R_2\, s}{s} \\ &\simeq \frac{R_2^2}{s} \qquad s \ll R_2 \\ &= \frac{4\pi R_2^2}{4\pi s} \\ &= \frac{A}{4\pi s}\ . \end{aligned}$$

On the second to last line, we've used the fact that $4\pi R_2^2$ is the surface area of the sphere.

6.3 Energy stored in a capacitor

Suppose we are charging up a capacitor. At some intermediate point in the process, there is a charge $+q$ on one plate, $-q$ on the other. We move a charge dq from the negative charge to the positive charge. It takes work to do this: we are forcing a positive charge in the direction opposite to what it "wants" to do. The amount of work we do moving this charge is

$$\begin{aligned} dW &= V(q)\, dq \\ &= \frac{q}{C} dq\ . \end{aligned}$$

Integrating this to a total charge separation of Q, we find that the work that is done charging up the capacitor is

$$\begin{aligned} W &= \int_0^Q \frac{q}{C} dq \\ &= \frac{1}{2}\frac{Q^2}{C}\ . \end{aligned}$$

This is the energy stored that is stored in the capacitor:

$$U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2} C V^2\ .$$

We've used $Q = CV$ in the second equality.

It's useful to carefully re-examine this formula for the case of the parallel plate capacitor: we put $C = A/4\pi s$, and note that the charge $Q = EA/4\pi$, where E is the electric field between the plates. Then,

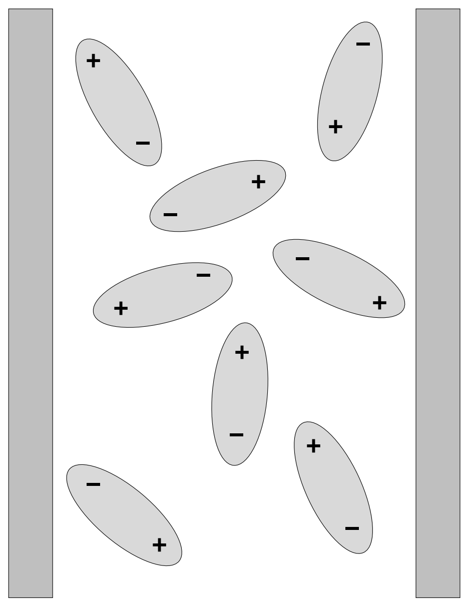
$$\begin{aligned} U &= \frac{1}{2}\frac{Q^2}{C} \\ &= \frac{1}{2}\frac{E^2 A^2}{16\pi^2}\frac{4\pi s}{A} \\ &= \frac{E^2}{8\pi} A s\ . \end{aligned}$$

The last line is the energy density of the electric field between the plates times the volume of that region.

6.4 Dielectrics

The capacitors we've discussed here are somewhat artificial: the space between their plates is taken to be empty. In the real world, the normal situation is that this space is filled with something — air, plastic, paper, compressed leprechauns, whatever. The "stuff" that fills this space interacts with the electric field, and can have an important impact on the the properties of the capacitor.

The basic idea can be understood fairly simply. The key point is that many materials are *polarizable*: they are made out of molecules that, though neutral overall, have a net excess of + charge at one end and an excess of − charge at the other. This means that each molecule looks like a little electric "dipole" — a distribution whose total charge is zero, but that shows non-trivial charge separation. Normally, the polar character of these molecules is unimportant — the random, mostly thermal motion of the molecules means that at any moment there is no net polarization to a big mass of them. If we take a block of such material and slide it between two parallel plates, it looks (roughly) like this:

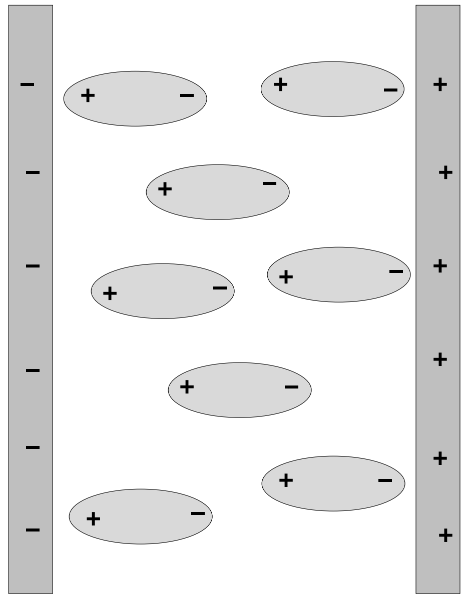


Suppose we now charge up these plates, so that there is an electric field between them. We leave these plates with *fixed charge*. All of these little dipoles will now want to line up — the + ends will align with the negatively charged plate, and vice versa:

By lining up in this way, they will tend to *reduce* the electric field between the plates — the field due to the charges on the plates themselves is superposed upon a field pointing in the opposite direction due to the alignment of the dipoles. For an enormous number of molecules, the reduction in the applied electric field can be summarized using a single number:

$$\text{Fixed charge situation:} \qquad E^{\text{with dielectric}} = E^{\text{without dielectric}}/K\ .$$

The number K is called the *dielectric constant*; it's something we can measure and catalog for various substances. Table 10.1 of Purcell lists a whole bunch; the values range from $K \simeq 1$ for air to $K \simeq 80$ for water. (Water has a huge dielectric constant because of the charge distribution associated with its bent molecules.) For us, the value of K is always greater than or equal to one.



Since the electric field is reduced by K, the voltage between the plates is likewise reduced. For the parallel plate capacitor and in cgs units, we have

$$V^{\text{with dielectric}} = \frac{4\pi Q s}{KA} .$$

Invoking the definition of the capacitance, we find

$$C^{\text{with dielectric}} = \frac{KA}{4\pi s} = KC^{\text{without dielectric}} .$$

It's worth thinking for a moment about what the dielectric does in some specific circumstances. Suppose I have a capacitor C whose plates have been charged up to some *fixed charge* Q. This is easily done by hooking a capacitor up to a voltage source V (e.g., a battery) and then removing the source — the capacitor will then hold the charge $Q = CV$. If I insert a dielectric between the plates, the capacitance increases; since the charge is fixed, the voltage must decrease:

$$\text{Fixed charge:} \qquad V^{\text{new}} = \frac{Q}{C^{\text{new}}} = \frac{Q}{KC^{\text{orig}}} = \frac{V^{\text{orig}}}{K} .$$

Suppose instead I take the capacitor and hold it at *fixed voltage*. This is also easy to do — we hook up our capacitor to a voltage source and just leave it there. The capacitance again increases, which means the charge must increase:

$$\text{Fixed voltage:} \qquad Q^{\text{new}} = C^{\text{new}}V = KC^{\text{orig}}V = KQ^{\text{orig}} .$$

The "extra" charge is actually pulled from the voltage source. Note that *in this circumstance* the electric field between the plates does not decrease when we insert the dielectric! However, it takes *more charge* to hold this field than it does without the dielectric — the dielectric "wants" to reduce the electric field; the voltage source needs to supply more charge to "fight" this tendency.

To wrap this section up, consider the capacitance in SI units. With a dielectric, it is given by

$$\begin{aligned} C^{\text{with dielectric}} &= \frac{K\epsilon_0 A}{s} \\ &\equiv \frac{\epsilon A}{s} \, . \end{aligned}$$

The combination $K\epsilon_0 \equiv \epsilon$ is called the *permittivity* of the dielectric. When $K = 1$, the permittivity is just ϵ_0, the permittivity of "free space" (meaning "empty space").

In practice, people don't usually describe dielectrics using permittivity — the dielectric constant K is much more useful. However, you do encounter the term sometimes; and, it helps to explain why ϵ_0 is called the permittivity of free space.