Scott Hughes *24 February 2005*

Massachusetts Institute of Technology
Department of Physics
8.022 Spring 2005

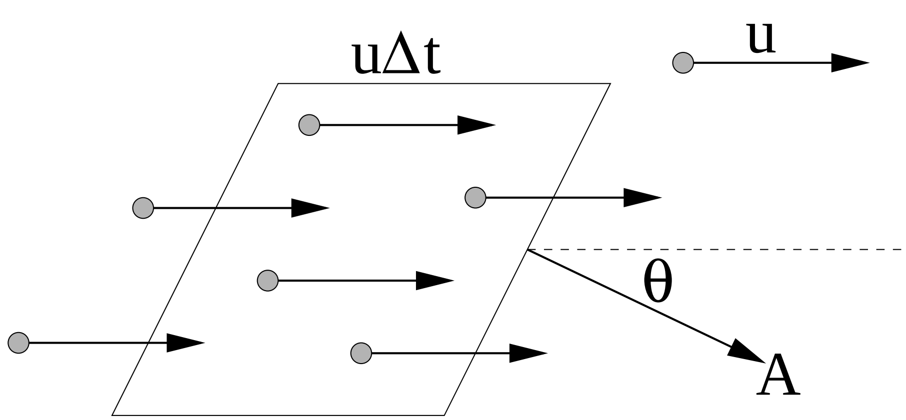
Lecture 7:
Current, continuity equation, resistance, Ohm's law.

7.1 Electric current: basic notions

The term "electric current" is used to describe the charge per unit time that flows through a region. In cgs units, current is measured in esu/sec, naturally enough. In SI units, current is measured in Coulombs/sec, which is given the name *Ampere* (or amp). It is very important to know how to convert between these units! Amps are used to describe currents *FAR* more often than esu/sec: 1 ampere $= 2.998 \times 10^9$ esu/sec.

Suppose we have a swarm of charges, all with the same charge q. The *number density* of these charges is some value n (i.e., there are n charges per unit volume). Suppose further that all of these charges are moving with velocity $\vec{u}$. How much current is flowing through an area $\vec{A}$?

To figure this out, we need to calculate how many of the charges pass through the area $\vec{A}$ in time Δt. This number is given by the number of charges that fit into the oblique prism sketched below:



The number of charges in this prism is its volume, $|\vec{A}||\vec{u}\Delta t|\cos\theta$, times the number density. The number of charges that pass through the area in time Δt is

$$\Delta N = n(|\vec{u}|\Delta t)(|\vec{A}|)\cos\theta = n\vec{u}\cdot\vec{A}\Delta t\ .$$

The total *charge* that passes through is q times this number:

$$\Delta q = qn\vec{u}\cdot\vec{A}\Delta t\ ,$$

so that the current $I = \Delta q/\Delta t$ must be

$$I = qn\vec{u}\cdot\vec{A}\ .$$

This form of the current motivates the definition of the *current density* $\vec{J}$:

$$\vec{J} = qn\vec{u} = \rho\vec{u} \ .$$

For this (admittedly artificial) example of all charges streaming in the same direction with the same velocity $\vec{u}$, the current density is just the charge density $\rho \equiv nq$ times that velocity. The current is then given by $I = \vec{J} \cdot \vec{A}$.

Caution: bear in mind that the current density is a current *per unit area*: charge/(time $\times$ area). This is hopefully obvious by dimensional analysis: charge density is charge/(length cubed); velocity is length/time; hence current density is charge/(time $\times$ length squared).

7.2 Electric current: more details

Some of the details assumed above are obviously fairly artificial. We make them more realistic one by one. First, rather than having a single kind of charge that is free to move, a material might have a bunch of different charges that can carry current. For example, ocean water conducts electricity largely because of the freely moving sodium and chlorine ions. Other materials dissolved in the ocean contribute a bit as well. Each variety of charge may have its own charge, number density, and velocity. Let the subscript k label different charge "species" — e.g., $k = 1$ could refer to electrons, $k = 2$ to chlorine ions, etc. The total current density comes from combining them all together:

$$\vec{J} = \sum_k q_k n_k \vec{u}_k = \sum_k \rho_k \vec{u}_k \ .$$

Note, via this formula, the equivalence between positive and negative charges we get by flipping the velocity direction: a positive charge moving with $\vec{u}_k$ contributes the same amount to the overall current density as a negative charge moving with $-\vec{u}_k$.

Next, the assumption that *all* charges of species k are moving with the exact same velocity $\vec{u}_k$ is pretty obviously ludicrous. A current of 1 amp corresponds to (for example) 6.24×10^{18} electrons per second streaming past. There's no way every single one of those electrons has the **exact** same velocity! So, in our formula for $\vec{J}$, we replace $\vec{u}_k$ with its value averaged over all the charges:

$$\langle \vec{u}_k \rangle = \frac{1}{N_k} \sum_{i=1}^{N_k} (\vec{u}_k)_i \ ,$$

where N_k is the total number of charge carriers of species k, and $(\vec{u}_k)_i$ is the velocity of the ith charge in species k (admittedly, not the sanest looking equation in the universe). The current density is thus given by

$$\vec{J} = \sum_k q_k n_k \langle \vec{u}_k \rangle = \sum_k \rho_k \langle \vec{u}_k \rangle \ .$$

(Note: Purcell gives a somewhat different formula for $\vec{J}$ built from the averaged velocity. I find his formula to be rather confusing; it took me half an hour to connect what he does with my own understanding of current density and averages. De mortuis nil nisi bonum.)

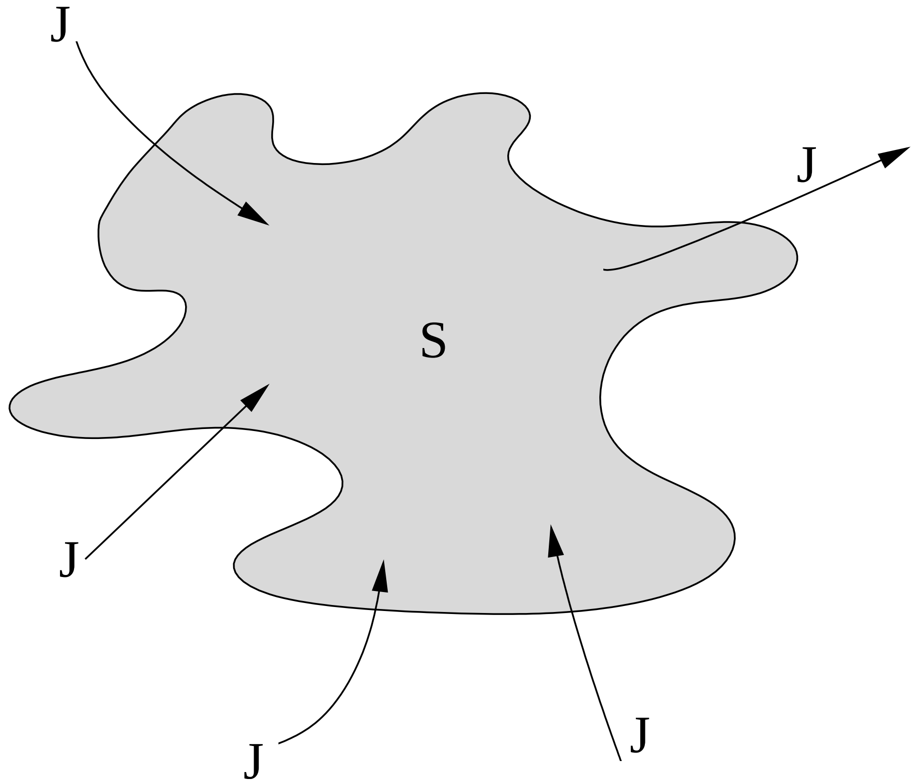
Finally, we do not expect in general to encounter nice, simple, uniform surfaces through which our current flows. Not surprisingly, we generalize to an integral: the current — charge/time — passing through some general surface S is given by

$$I = \int_S \vec{J} \cdot d\vec{A} \ .$$

We can thus think of total current as the flux of current density through a surface.

7.3 Charge conservation and continuity

Suppose the current flows through a *closed* surface:



In this case, there *must* be charge piling up inside (or departing from) the volume V enclosed by the surface S:

$$\oint_S \vec{J} \cdot d\vec{A} = -\frac{\partial}{\partial t} Q(\text{contained by } S) \ .$$

The minus sign in this equation comes from our convention on assigning direction to the area element $d\vec{A}$ — since $d\vec{A}$ points out, $\vec{J} \cdot d\vec{A}$ is negative if the current density points inward, and is positive if the current density points outward. Since inward corresponds to a gain of charge (and vice versa) we clearly need a minus sign for the physics to make sense.

Using our skill at vector calculus, we can manipulate this equation as follows:

$$\oint_S \vec{J} \cdot d\vec{A} = \int_V \left(\vec{\nabla} \cdot \vec{J}\right) dV \ ,$$

where V is the volume enclosed by S. This is just Gauss's theorem. We also have

$$Q(\text{enclosed by } S) = \int_V \rho \, dV \ .$$

Putting this together we have

$$\begin{aligned}
\int_V \left(\vec{\nabla} \cdot \vec{J}\right) dV &= -\frac{\partial}{\partial t} \int_V \rho \, dV \\
&= -\int_V \frac{\partial \rho}{\partial t} dV \\
\rightarrow \int_V \left(\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t}\right) dV &= 0 \ .
\end{aligned}$$

This equation is guaranteed to hold only if

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \ .$$

This result is known as the *continuity equation*: it guarantees conservation of electric charge in the presence of currents. It also tells us that when currents are *steady* — no time variation, so that $\partial\rho/\partial t = 0$ — we must have $\vec{\nabla} \cdot \vec{J} = 0$.

7.4 Ohm's law

Electric fields cause charges to move. It stands to reason that an electric field applied to some material will cause currents to flow in that material.

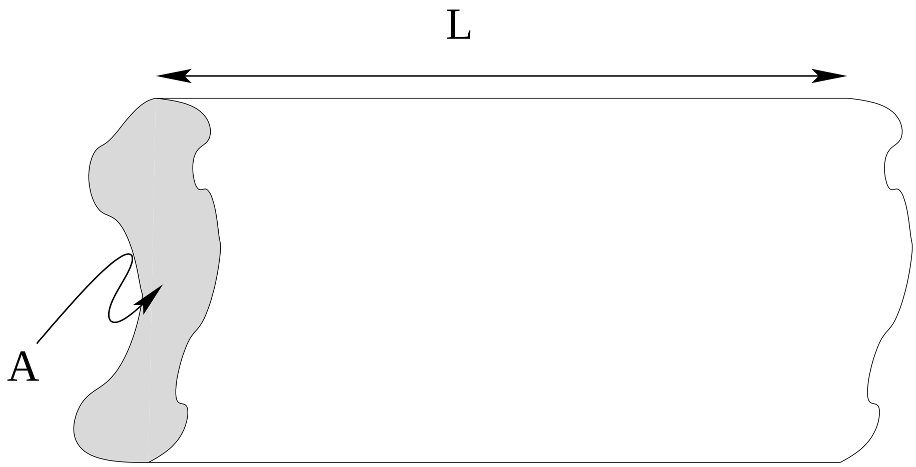
For an amazingly wide range of materials, an empirical rule called Ohm's law gives the following relation between current density and applied electric field:

$$\vec{J} = \sigma \vec{E} \ .$$

In other words, the current density is directly proportional to the electric field. The constant of proportionality σ is called the material's *conductivity*.

CAUTION: the symbol σ stands for BOTH conductivity AND surface charge density!!!! This can be annoying, but is a convention that we are stuck with, sadly. You should always know from the context of a given equation the physical meaning of a particular σ. Even more sadly, there are times when both meanings of σ need to be used simultaneously. In such a case, I suggest using subscripts to distinguish the two symbols — e.g., σ_c gives the conductivity, σ_q is surface charge density.

Ohm's law comes in two flavors. The version already given, $\vec{J} = \sigma\vec{E}$, can be thought of as the "local" or "microscopic" version. It tells me about the relation between current density and field at *any given point* in some material. To understand the second version, consider current flowing in some chunk of material. The material is perfectly homogeneous, meaning that the conductivity is σ throughout the chunk. It is of length L, and has a cross-section area A:



We put this material in a *uniform* electric field $\vec{E}$. Let's orient this field parallel to the material's long axis. Then, the potential difference or "voltage" between its two ends is $V = EL$.

At every point in the chunk's interior, Ohm's law holds: $J = \sigma E$ (dropping the vector signs since everything points along the long axis). Since E is constant everywhere, and since the chunk is homogeneous, this value of J holds everywhere. The *total* current flowing past any point must be $I = JA$. We now look for a relationship between the current I and the voltage V:

$$\begin{aligned} J &= \sigma E \\ \frac{I}{A} &= \sigma \frac{V}{L} \\ \rightarrow V &= I\left(\frac{L}{\sigma A}\right) \\ &\equiv IR\,. \end{aligned}$$

The formula $V = IR$ is the "global" or "macroscopic" form of Ohm's law: it is entirely equivalent to $J = \sigma E$, but involves quantities which are integrated over the macroscopic extent of our conducting material. The quantity in parentheses is defined as the *resistance* R of the chunk of material:

$$R \equiv \frac{L}{\sigma A} \equiv \frac{\rho L}{A}\,.$$

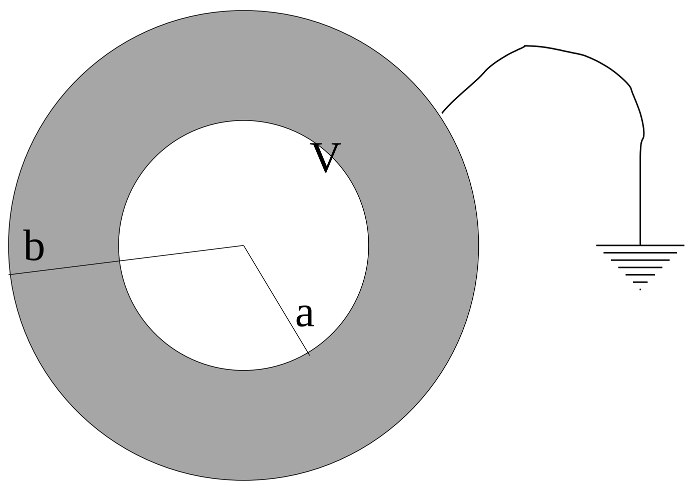
In this context, the symbol $\rho \equiv 1/\sigma$ is the *resistivity* of the material. Sadly, this is yet another example of overusing symbols — be careful not to confuse ρ as resistivity with ρ as charge density. The key thing to note about the resistance is that it goes proportional to the length of a material, and inversely proportional to the cross-sectional area.

Units: In SI units, resistance is measured in Ohms, which is just a special name for Volts/Amperes. In cgs units, the voltage is measured in esu/cm, and the current in esu/sec, so the resistance is measured in sec/cm, a rather odd unit. From this, we can infer the units of resistivity and conductivity: resistivity is measured in Ohm-meters in SI units[1], and in seconds in cgs.

7.4.1 Example: Resistance of a spherical shell

We derived the formula for the resistance of a piece of material for a very simple geometry. Similar calculations apply even if things are arranged in a goofier way.

Suppose we have a pair of nested spheres; the space between them is filled with material whose resistivity is ρ. The inner sphere is held at constant potential V; the outer sphere is *grounded*, so that its potential is zero. The radius of the inner sphere is a; that of the outer sphere is b. The current flows steadily — there is no charge piling up anywhere. What is the resistance of this shell?



We want to use Ohm's law in microscopic form, $J = \sigma E$, express the field in terms of V, the current density in terms of a current I, and then read off the resistance R. To do that, we need to know the potential at any point between the plates. By spherical symmetry, the potential must clearly have the form $\phi = C/r + D$, for some constants C and D. We have the boundary conditions that at $r = a$, $\phi = V$; at $r = b$, $\phi = 0$:

$$\begin{aligned} V &= \frac{C}{a} + D \\ 0 &= \frac{C}{b} + D \, . \end{aligned}$$

With a bit of effort, we solve these equations to find

$$\begin{aligned} C &= V\frac{ab}{b-a} \\ D &= -V\frac{a}{b-a} \, . \end{aligned}$$

[1] In practice, this unit is rarely used. The unit Ohm-cm is used much more often.

Now, we use $E = -\partial\phi/\partial r$ to find

$$E = \frac{V}{r^2}\frac{ab}{b-a} .$$

So, we now know the electric field. Note this points radially.

How much current passes through a shell of radius r somewhere between a and b? We must have

$$I = 4\pi r^2 J .$$

Note that this current must be constant! There is no charge piling up, so the same total charge per unit time passs through any given shell of radius r.

Now, we impose the microscopic form of Ohm's law:

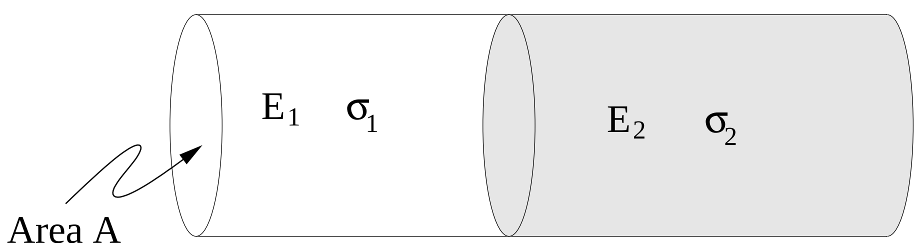
$$\begin{aligned} I &= (4\pi r^2)(\sigma E) \\ &= (4\pi r^2)\sigma\frac{V}{r^2}\frac{ab}{b-a} \\ &= \frac{4\pi\sigma ab}{b-a}V . \end{aligned}$$

Rearranging to put this in the form $V = IR$, putting $\rho = 1/\sigma$, we read off the resistance of the shell:

$$R = \frac{\rho(b-a)}{4\pi ab} .$$

7.4.2 Example: Variation of σ

What happens if σ is not a constant? The answer is set by requiring that the current flow be steady. Consider two conductors of conductivity σ_1 and σ_2 fused together:



The current must be the same in each half:

$$I = JA = \sigma_1 E_1 A = \sigma_2 E_2 A .$$

This can only be true if the electric field is different in each material. The electric field thus suddenly jumps as we cross from one side of the junction to the other, which means that there must be a layer of surface charge σ_q at the boundary[2]:

$$\sigma_q = \frac{1}{4\pi}(E_2 - E_1) = \frac{I}{4\pi A}\left(\frac{1}{\sigma_2} - \frac{1}{\sigma_1}\right) .$$

Whenever the conductivity varies, there is a possibility that, in the steady state, some charge accumulates somewhere. This is necessary to maintain steady current flow.

[2] Here's where the overuse of the symbol σ really gets annoying. Sorry.

7.5 A closer look at Ohm's law

Ohm's law, on the face of it, makes no sense. It tells us that under an applied electric field, we get a steady current density. Since $\vec{J} = qn\vec{u}$ (considering only one current carrier), this means that the electric field is causing currents to move at constant velocity.

This should bug you! Electric fields cause *forces*; forces cause *accelerations*. Why don't the charges just accelerate like mad?

The answer is that the charges are in a medium — the conductor — that prevents them from moving freely. The motion is essentially like that in a viscous medium. The situation is much closer to a sky diver in free fall — after a short period of acceleration, the charges reach "terminal velocity", and essentially ride along at that velocity from then on. This "terminal velocity" is what appears in the current density formula.

The cause of this effectively viscous motion is (roughly) the many collisions that the charge carriers experience as they flow through the conductor. The charge carriers accelerate like hell for a short time, smash into something, get directed in some random direction, charge like hell again, and repeat[3]. The net *averaged* behavior is a uniform drift.

More concretely, let's look at how the momentum of charge carriers accumulate in a conductor. Consider a single charge first. Suppose at $t = 0$ a collision just occurred. Its momentum following this collision $\vec{p} = m\vec{u}_0$, where $\vec{u}_0$ is the random velocity it has following the collision. The electric field acting on this charge changes its momentum over a time interval t by imparting an impulse $\Delta\vec{p} = q\vec{E}t$. The total momentum of this particle, $\vec{p} = m\vec{u}_0 + q\vec{E}t$, thus grows linearly with time until the next collision occurs, at which point the clock is reset and we repeat.

Now consider a whole swarm of N of these charges. The *average* momentum of a member of this ensemble is

$$m\langle\vec{u}\rangle = \frac{1}{N}\sum_{i=1}^{N}\left(m\vec{u}_i + q\vec{E}t_i\right) \ .$$

The velocity $\vec{u}_i$ is the random velocity imparted to charge i following its last collision; t_i is the time interval for that charge since that collision. If N is a large number, the first term *must be extremely close to zero.* Why? We're adding up a enormous number of randomly oriented vectors! The result of such addition is always a number that is very, very close to zero, provided N is large[4]. What remains is

$$m\langle\vec{u}\rangle = \frac{q\vec{E}}{N}\sum_{i=1}^{N} t_i \equiv q\vec{E}\tau \ ,$$

where

$$\tau \equiv \frac{1}{N}\sum_{i=1}^{N} t_i$$

is the average time between collisions. This turns out to be a property of the material; this, then, is what causes an imposed electric field to lead to uniform velocity.

[3]If you've ever driven through Los Angeles during rush hour, you've experienced the life of a charge carrier in a conductor.

[4]A careful analysis of this kind of addition shows that there is some error incurred by assuming that the vectors sum to precisely zero. However, that error is of order $1/\sqrt{N}$. As long as N is very large — which is the case for any good conductor, where N is of order Avogadro's number — these small errors are not important. Those of you who take 8.044 will learn more about this.

Notice that from this derivation, we can just read off the conductivity:

$$\begin{aligned}\vec{J} &= nq\langle\vec{u}\rangle = nq\left(\frac{q\vec{E}\tau}{m}\right) = \sigma\vec{E} \\ \rightarrow \sigma &= \frac{nq^2\tau}{m}\ .\end{aligned}$$

If we have multiple charge carriers, we find

$$\sigma = \sum_k \frac{n_k q_k^2 \tau_k}{m_k}\ .$$