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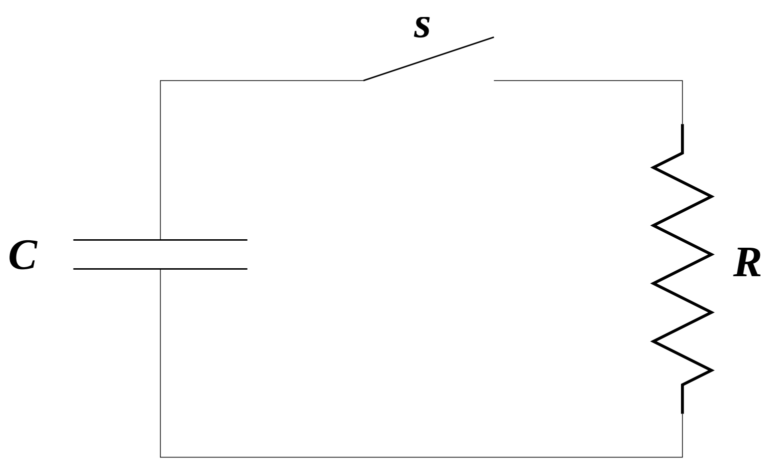
Lecture 9:
Variable currents; Thévenin equivalence

9.1 Variable currents 1: Discharging a capacitor

Up til now, everything we have done has assumed that things are in steady state: all fields are constant; charges are either nailed in place or else are flowing uniformly. In reality, such a situation tends to be the exception rather than the rule. We now start to think about things that vary in time.

Suppose we have a capacitor C charged up until the potential difference between its plates is V_0; the charge separation is $Q_0 = CV_0$. We put this capacitor into a circuit with a resistor R and a switch s:



What happens when the switch is closed?

Before thinking about this with equations, let's think about what happens here physically. The instant that the switch is closed, there is a potential difference of V_0 across the resistor. This drives a current to flow. Now, this current can only come from the charge separation on the plates of the capacitor: the excess charges on one plate flow off and neutralize the deficit of charges on the other plate. The flow of current thus serves to reduce the amount of charge on the capacitor; by $Q = CV$, this must reduce the voltage across the capacitor. The potential difference which drives currents thus becomes smaller, and so the current flow should reduce. We expect to see a flow of current that starts out big and gradually drops off.

To substantiate this, turn to Kirchhoff's laws: at any moment, the capacitor supplies an EMF $V = Q/C$. As the current flows, there is a voltage drop $-IR$ across the resistor:

$$\frac{Q}{C} - IR = 0\ .$$

This is true, but not very helpful — we need to connect the charge on the capacitor Q to the flow of current I. Thinking about it a second, we see that we must have

$$I = -\frac{dQ}{dt} .$$

Why the minus sign? The capacitor is *losing* charge; more accurately, the charge separation is being reduced. In this situation, a large, positive current reflects a large reduction in the capacitor's charge separation.

Putting this into Kirchhoff, we end up with a first order differential equation:

$$\frac{Q}{C} + R\frac{dQ}{dt} = 0 .$$

To solve it, rearrange this in a slightly funny way:

$$\frac{dQ}{Q} = -\frac{dt}{RC} .$$

Then integrate both sides. We use the integral to enforce the *boundary conditions*: initially ($t = 0$), the charge separation is Q_0. At some later time t, it is a value $Q(t)$. Our goal is to find this $Q(t)$:

$$\int_{Q=Q_0}^{Q=Q(t)} \frac{dQ}{Q} = -\int_{t=0}^{t} \frac{dt}{RC}$$

$$\rightarrow \ln\left[\frac{Q(t)}{Q_0}\right] = -\frac{t}{RC} .$$

Taking the exponential of both sides gives the solution:

$$Q(t) = Q_0 e^{-t/RC} .$$

The charge decays exponentially. After every time interval of RC, the charge has fallen by a factor of $1/e$ relative to its value at the start of the interval.

Sanity check: does RC make sense as a time? Let's check its units:

- CGS: $R \times C$ has units (sec/cm) $\times$ (cm) = seconds.
- SI: $R \times C$ is (ohms) $\times$ (farads) = (volts/amps) $\times$ (coulombs/volts) = coulombs/amps = seconds.

So it does make sense.

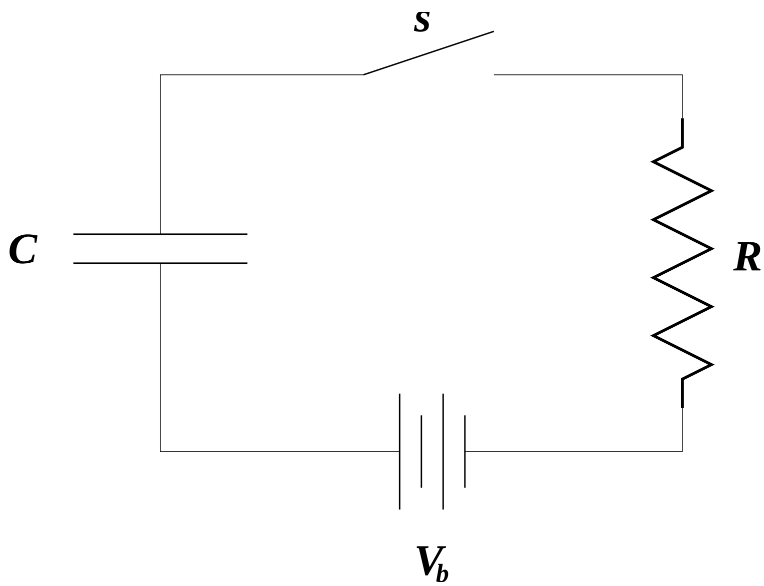
Let's look at the current $I(t)$:

$$I(t) = -\frac{dQ}{dt} = -Q_0\frac{d}{dt}e^{-t/RC} = \frac{Q_0}{RC}e^{-t/RC} .$$

Just like the charge, the current decays exponentially. In particular, it starts out at some big value, and then falls away — just as our physical intuition told us it would.

9.2 Variable currents 2: Charging a capacitor

Let's consider a different kind of circuit. The capacitor begins, at $t = 0$, with *no* charge; but, the circuit now contains a battery:



Now, what happens when the switch is closed?

Let's again think about this physically before doing the math. When the switch is first closed, the EMF of the battery can very easily drive a current, so we expect the current to jump up, and charge to begin accumulating on the capacitor. As this charge accumulates, an $\vec{E}$ field will build up in a direction that *opposes* the flow of current! Thus, we expect that, as time goes on, the flow of current will decrease. Eventually it will stop, when there is just enough charge on the capacitor to totally oppose the battery.

Let's check this. For Kirchhoff, we have an EMF V_b from the battery, and two voltage drops: $-Q/C$ from the capacitor, and $-IR$ across the resistor:

$$V_b - \frac{Q}{C} - IR = 0 \; .$$

We again need to relate the current I and the capacitor's charge Q. Thinking about it for a second, we must have

$$I = +\frac{dQ}{dt} \; .$$

Why a plus sign? In this case we are *adding* charge to the capacitor (more accurately, *increasing* the charge separation). Large current means a large increase in the charge separation. When doing a problem involving a circuit like this, it is important to stop and think carefully about how the capacitor's charge relates to the charge flowing onto it or off of it.

We end up with the following differential equation:

$$V_b - \frac{Q}{C} - R\frac{dQ}{dt} = 0 \; .$$

Rearrange:

$$\frac{dQ}{dt} = -\frac{Q - CV_b}{RC}$$

or

$$\frac{dQ}{Q - CV_b} = -\frac{dt}{RC} \ .$$

Integrate: we require $Q(t = 0) = 0$, so we find

$$\ln\left[\frac{CV_b - Q(t)}{CV_b}\right] = -\frac{t}{RC} \ .$$

Exponentiating both sides and solving for $Q(t)$, we find

$$Q(t) = CV_b\left(1 - e^{-t/RC}\right) \ .$$

This charge starts off at zero, builds up quickly at first, then starts building up more and more slowly. It asymptotically levels off at $Q = CV_b$ as $t \to \infty$. Let's look at the current:

$$\begin{aligned} I(t) &= CV_b\frac{d}{dt}\left(1 - e^{-t/RC}\right) \\ &= \frac{V_b}{R}e^{-t/RC} \ . \end{aligned}$$

As we intuitively guessed, the current starts large, but drops off as time passes and the capacitor's electric field impedes it.

Notice that the voltage across the capacitor

$$V_C(t) = \frac{Q(t)}{C} = V_b\left(1 - e^{-t/RC}\right)$$

and the voltage across the resistor

$$V_R(t) = I(t)R = V_b e^{-t/RC}$$

sum to give a constant,

$$V_C(t) + V_R(t) = V_b\left(1 - e^{-t/RC}\right) + V_b e^{-t/RC} = V_b \ ,$$

the EMF of the battery. If you think about Kirchhoff's rules for a moment, this should make a lot of sense!

9.3 Charging and discharging revisited

Suppose you're taking a test and you don't want to slog through setting up a bunch of integrals and running the risk of botching the details because of a silly mistake. Provided you understand *physically* what is going on in the above discussions, you can write down the rules for charge on a capacitor without doing any math. You only need to know three things:

1. The time variation has to look like $e^{-t/RC}$.
2. You need the initial ($t = 0$) charge on the capacitor, and
3. You need the final ($t \to \infty$) charge on the capacitor.

Then, you just need to figure out to hook everything together so that you get a formula that obeys the correct boundary conditions. For example, if a problem involves discharging a capacitor, you know that you start out with some initial value Q_0, and that it must fall towards zero as time passes. The only formula that obeys these conditions and has the correct time variation is

$$Q(t) = Q_0 e^{-t/RC} ,$$

just what we derived carefully before. If it involves charging up a capacitor, you want a formula that has $Q = 0$ at $t = 0$, and that levels off at CV_b as $t \to \infty$. Using $1 - e^{-t/RC}$ as our "time variation" piece of the solution gets the $t = 0$ behavior right; multiplying by CV_b ensures that we level off at the right value at large t. We end up with

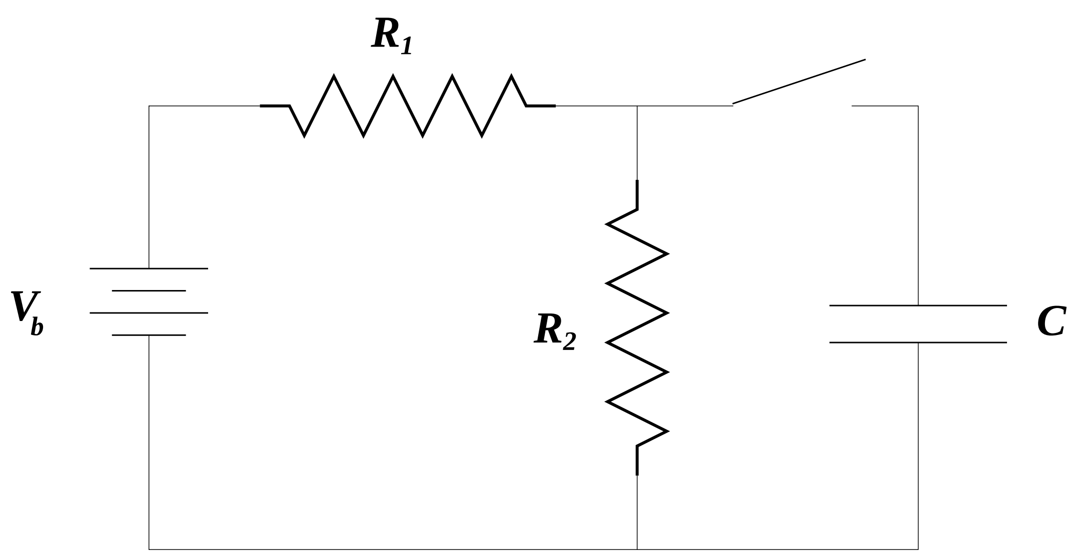
$$Q(t) = CV_b \left(1 - e^{-t/RC}\right) .$$

Again, just what we derived carefully before.

If you can remember these physical reasons why the charge behaves as it does, you can save yourself some pain later on.

9.4 Thévenin equivalence

The RC circuits we've looked at so far are quite simple — there's only one resistor, and it's hooked up to the capacitor and/or battery in series. How do we handle a more complicated circuit? Consider the following:



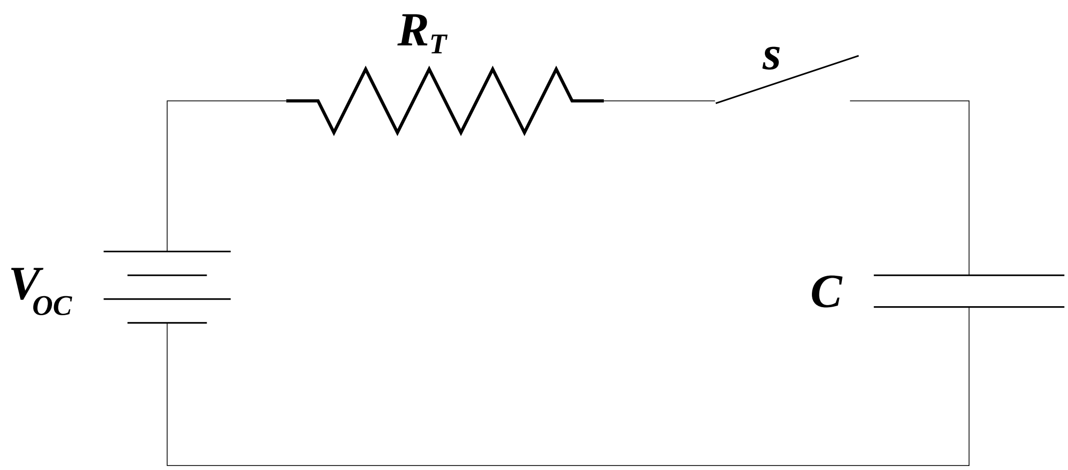
How does the charge on the capacitor evolve with time?

This problem can be tackled quite simply by using Kirchhoff's laws: we have multiple loops, so we go around them and write down equations balancing the voltage drops and EMFs; we make sure the currents balance at the junctions; math happens; we find $Q(t)$. As an exercise, this is straightforward, but slightly tedious.

There is in fact a very cute way to approach this, based on a theorem proved by Léon Charles Thévenin[1]:

Thévenin's Theorem: Any combination of batteries and resistances with two terminals can be replaced by a single voltage source V_{OC} and a single series resistor R_T.

For our problem, this means that we can redraw the circuit as



The answer is now obvious! We just steal results from before, and write down

$$Q(t) = CV_{\text{OC}}\left(1 - e^{-t/R_T C}\right) \ .$$

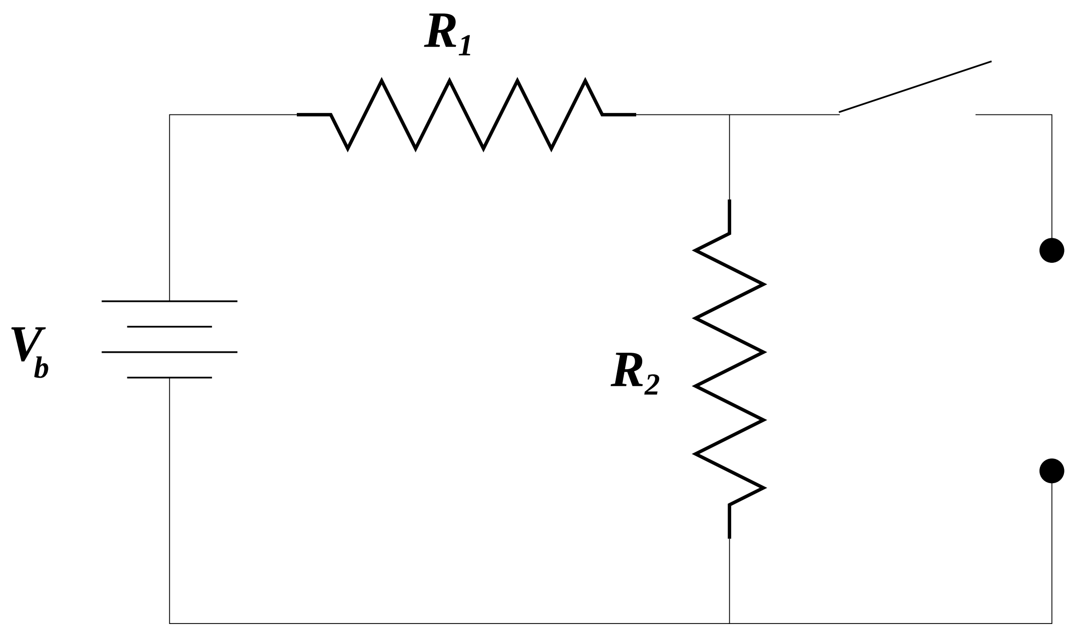
We just need to know how to work out V_{OC} and R_T.

[1] I tried to find biographical information about this guy and totally failed. Fortunately, an anonymous Spring '04 8.022 student was more persistant than me, and found a web bio:

`http://www.eleceng.adelaide.edu.au/famous.html#thevenin`

He was a French telegraph engineer, and hence very well versed in practical aspects of electrical circuits. Interestingly, "his" theorem was actually first derived by Hermann Von Helmholtz.

To do this, we take the capacitor out of the circuit and consider what remains as a pair of terminals:



If we short circuit these terminals with a wire, a current $I_{\rm SC}$ — the "short circuit" current — will flow. On the other hand, if we leave it open, there will be a potential difference — the "open circuit" voltage, $V_{\rm OC}$ — between the terminals (equivalent to the potential drop over the resistor R_2).

Let's calculate these quantities. Shorting the terminals out is equivalent to putting a 0 Ohm resistor in parallel with R_2. The current that flows is what you would get if R_2 were not even present:

$$I_{\rm SC} = \frac{V_b}{R_1} \ .$$

On the other hand, if we leave the terminals open, the current that flows through the circuit is $I = V_b/(R_1 + R_2)$. The open circuit voltage is thus given by

$$V_{\rm OC} = V_{R_2} = IR_2 = \frac{V_b R_2}{R_1 + R_2} \ .$$

The *Thévenin equivalent resistance* is defined as the ratio of these quantities:

$$R_T \equiv \frac{V_{\rm OC}}{I_{\rm SC}} \ .$$

For this particular circuit, it takes the value

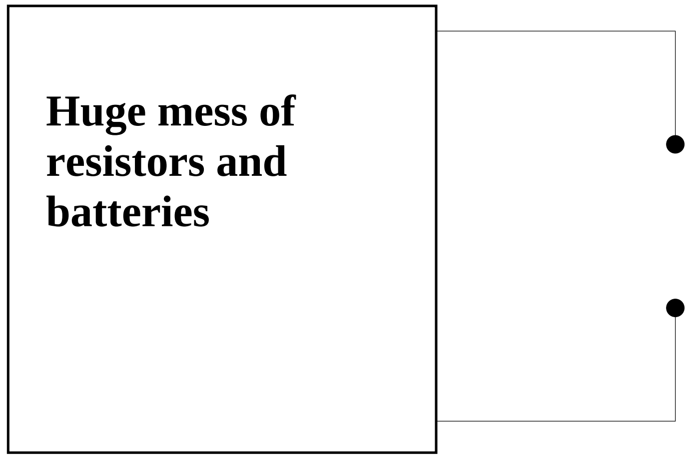
$$\begin{aligned} R_T &= \frac{V_b R_2}{R_1 + R_2} \times \frac{R_1}{V_b} \\ &= \frac{R_1 R_2}{R_1 + R_2} \ . \end{aligned}$$

The solution for the charge on the capacitor is thus finally given by

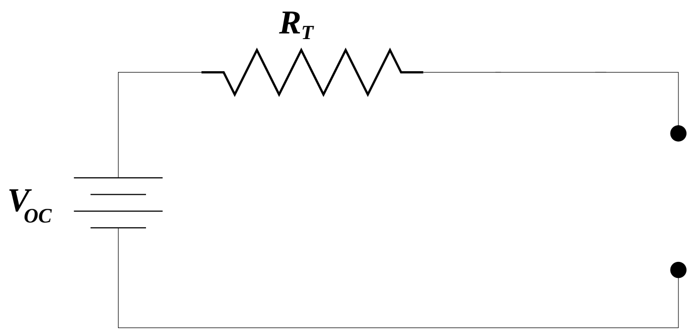
$$Q(t) = CV_b \left(\frac{R_2}{R_1 + R_2} \right) \left[1 - \exp(-t(R_1 + R_2)/CR_1R_2) \right] \ .$$

9.5 Thévenin in general

Thévenin's theorem is much more general than this simple example. In general, it means that you can be given any grungy mess of resistors and batteries and reduce it to a single battery with EMF V_{OC} and a single resistor R_T: anything like this



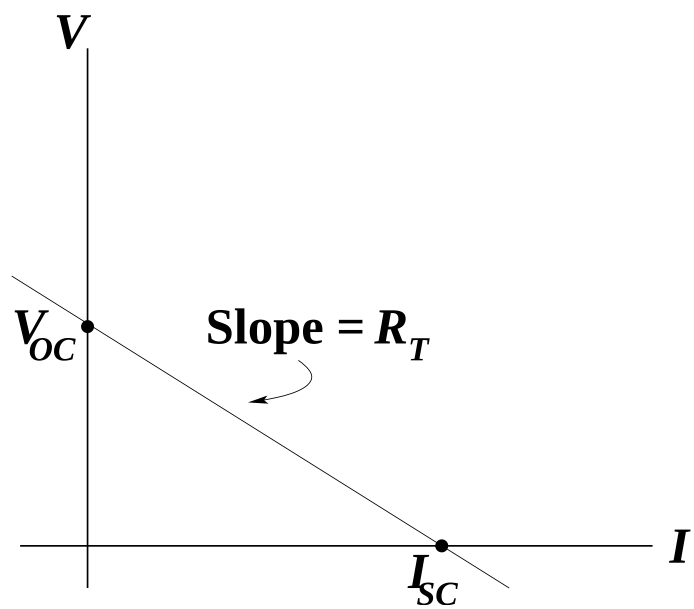
can be reduced to this



If you know what the big mess inside the "black box" is made out of, then you can figure out V_{OC} and R_T as we did for the example above.

In some situations, you *don't* know. For example, you might have a piece of lab equipment that you need to use for some experiment, and you need to know how to treat it in a circuit. In that case, you measure V_{OC}; you short out the terminals and measure I_{SC}; and then you know $R_T = V_{\text{OC}}/I_{\text{SC}}$.

Thévenin's theorem works only when all the elements inside the box obey Ohm's law. This means that the relationship between current and voltage for every element is *linear*. This in turn means that the relationship between current and voltage for any *combination* of circuit elements in the box must be linear. (Why? Add a bunch of lines together: you get a line!) The procedure for determining the Thévenin equivalence of a circuit simply assumes that you have some linear relationship between the voltage and current at the terminals. V_{OC} tells us where this relationship crosses the voltage axis; I_{SC} tells us where it crosses the current axis. R_T is the slope. (Negative slope, really, since the line rises in the "wrong" direction; same difference.)


V
V_OC
Slope = R_T
I
I_SC