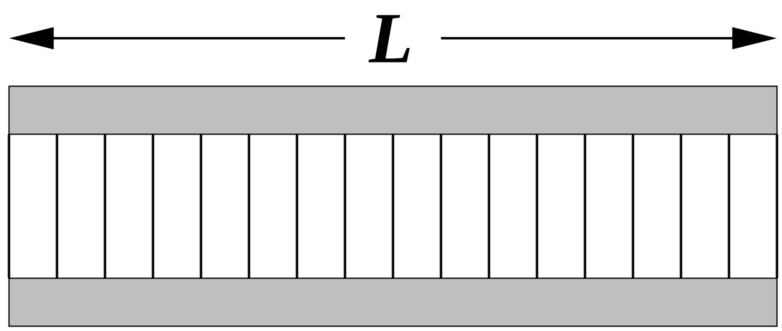
Massachusetts Institute of Technology
Department of Physics
8.022 Spring 2005

Lecture 12:
Forces and Fields in Special Relativity

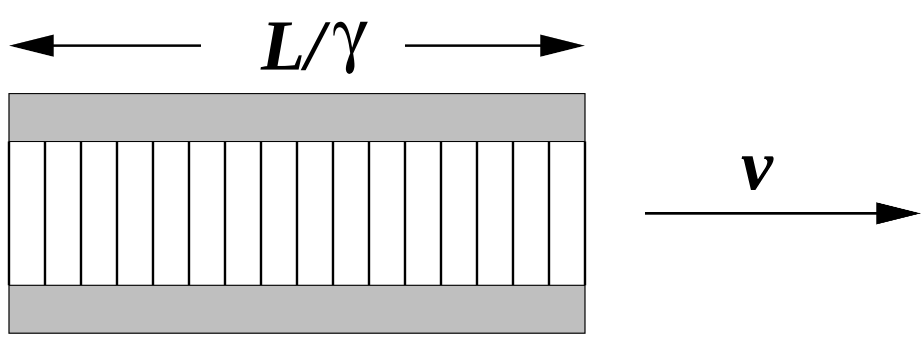
12.1 Transformation of the electric field

It is very easy to see how the electric field transforms by considering a special case. Examine a capacitor in its rest frame. Orient the capacitor such that its plates are parallel to the $x - y$ plane. We take the capacitor to have charge density σ on the lower plate, $-\sigma$ on the upper plate. We take the plates to be square, with each side having length L. The total amount of charge on the plates is thus $Q = \sigma L^2$; the electric field between the plates is $\vec{E} = 4\pi\sigma\,\hat{z}$:



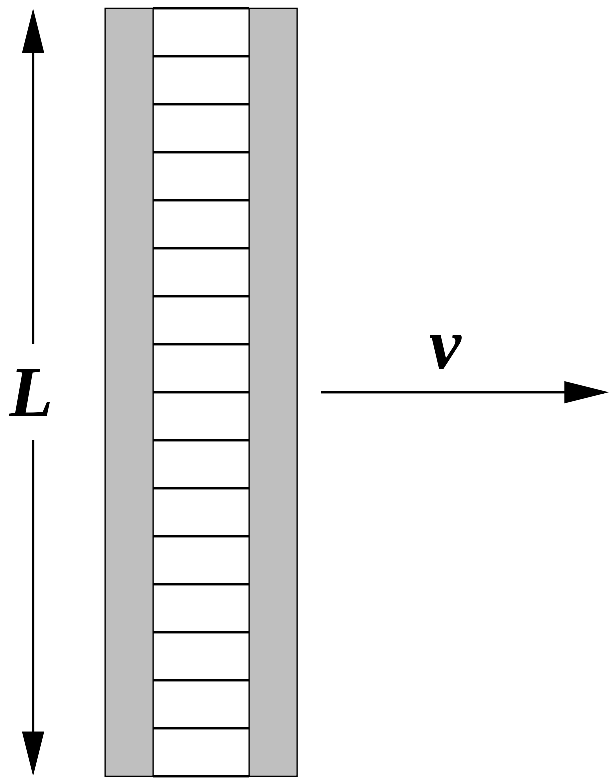
The field inside the capacitor is represented by the field lines sketched here.

Consider now this capacitor "boosted" to a velocity v in the x direction. Charge is a "Lorentz invariant": all observers agree that the total charge on the plates is Q. Because of the Lorentz contraction, the plate's dimension along the direction of motion is seen to be L/γ.



The charge density is thus augmented: $\sigma' = Q/(L \times L/\gamma) = \gamma Q/L^2 = \gamma\sigma$. The electric field is therefore augmented as well: $\vec{E}' = 4\pi\gamma\sigma\,\hat{z}$. Notice that the field lines are denser in this picture: keep in mind that density of field lines tells us how strong the field is.

How about if the capacitor is oriented differently? Let's rotate it so that the plates lie in the $y - z$ plane. The electric field in the rest frame is thus $\vec{E} = 4\pi\sigma\,\hat{x}$. When this capacitor is boosted in the x direction, the plates appear closer together, but the charge density *does not change*: $\sigma' = \sigma$ with this orientation: $\vec{E}' = \vec{E} = 4\pi\sigma$.



Notice that the density of field lines are not changed in this picture, reflecting the fact that the field is not changed by the boost.

Even though we have only looked at the electric fields for a rather special circumstance, the rule we deduce from this is in fact quite general: **Components of the electric field perpendicular to the velocity are augmented by the Lorentz transformation; components that are parallel to the velocity are unaffected**. Mathematically,

$$\begin{aligned} E'_\perp &= \gamma E_\perp \\ E'_{||} &= E_{||} \,. \end{aligned}$$

12.2 Transformation of energy and momentum

Our next goal will be to understand how forces transform between different frames of reference. To do this, we first must understand how to describe energy and momentum in special relativity.

Consider an object whose *rest mass* — the mass that we would measure when this object is at rest with respect to us — is m. If this object moves with velocity $\vec{u}$, what is its energy and its momentum? In the "old-fashioned" mechanics you came to know and love in 8.012, we said that a moving body has a kinetic energy

$$E_{\rm kin} = \frac{1}{2}m|\vec{u}|^2$$

and a momentum

$$\vec{p} = m\vec{u}\ .$$

These definitions form the foundations of kinematics — using them and the notions of *conservation of momentum* and *conservation of energy*, an enormous amount of mechanics follow.

To develop a generalization of kinematics for special relativity, we insist that some notion of "conservation of energy" and "conservation of energy" must be preserved — *some* kind of notion of energy and momentum must be the same before and after two objects collide (for example), even if those objects are moving at near light speed relative to one another. We unfortunately will not be able to go into the "hows" of the way that this works in this course; interested former 8.012 students can find good discussion in Chapter 13 of Kleppner and Kolenkow's textbook; students who take 8.033 next year will study this by the bucketload. I will instead just quote the final result: "conservation of energy" and "conservation of momentum" work in special relativity by requiring that the energy and momentum of a mass observed to move with velocity $\vec{u}$ is given by

$$\begin{aligned} \vec{p} &= \gamma_u m\vec{u} \\ E &= \gamma_u mc^2 \end{aligned}$$

where $\gamma_u = 1/\sqrt{1 - u^2/c^2}$. Before moving on, it is worth looking at the small u expansion of these quantities. We expand γ_u as

$$\gamma_u \simeq 1 + \frac{1}{2}\frac{u^2}{c^2} \qquad u \ll c\ .$$

Using this, we find

$$\begin{aligned} \vec{p} &\simeq \left(1 + \frac{1}{2}\frac{u^2}{c^2}\right) m\vec{u} \\ &\simeq m\vec{u} \\ E &\simeq \left(1 + \frac{1}{2}\frac{u^2}{c^2}\right) mc^2 \\ &\simeq mc^2 + \frac{1}{2}mu^2\ . \end{aligned}$$

In the momentum formula, we could have included another term whose magnitude scales proportional to u^3; for small u, this term is negligible and can usually be ignored. The result is nothing more than the "usual" momentum we all learned to love and cherish in 8.012.

The energy formula is particularly interesting: it tells us that the energy of a body with mass m is just the "usual" kinetic energy, $mu^2/2$, plus a "rest energy" mc^2. This is probably the most well-known physics equation on Earth, the energy associated with mass itself.

For our purposes, the most important result will be how these generalizations of energy and momentum Lorentz transform. Suppose in frame 1 — the "unprimed" frame — a body is seen to move with velocity $\vec{u}$. Its energy E and momentum $\vec{p}$ will then be given by the formulas above.

Consider now frame 2. Frame 2 claims that frame 1 is moving with velocity $\vec{v} = v\hat{x}$. What are the energy and angular momentum of the body as seen in frame 2 — the "primed" frame? The Lorentz transformation we need is given by

$$\begin{aligned}
E' &= \gamma_v(E - \beta_v c p_x) \\
p'_x &= \gamma_v(p_x - \beta_v E/c) \\
p'_y &= p_y \\
p'_z &= p_z
\end{aligned}$$

where $\beta_v = v/c$, and $\gamma_v = 1/\sqrt{1 - v^2/c^2}$.

12.3 Transformation of forces

We are now ready to work out how force transforms between reference frames. Let us first work out a few quantities in the "rest frame" of our body. Note that the name "rest frame" is somewhat misleading here — if there's a force acting on it, it must be accelerating, so it isn't going to be at rest for very long! Our "rest frame" will be the frame in which the body is *initially* at rest — $p = 0$ at $t = 0$, and then moves slowly enough that the old-fashioned, slow u description of the body's motion is accurate.

Suppose the force acts in the x direction. In this "rest frame", the force on the body is

$$F_x = \frac{dp_x}{dt} \, .$$

For what follows, we will also need to know about *changes* in the body's position and in its energy. The change in its position is given by

$$\Delta x = \frac{1}{2}\left(\frac{F_x}{m}\right)\Delta t^2$$

since F_x/m is the acceleration of the body. The change in its energy is given by

$$\Delta E = \frac{(F_x \Delta t)^2}{2m} \, .$$

You should be able to show very easily that this is just $\frac{1}{2}mu^2$ [bear in mind that $u = a\Delta t = (F_x/m)\Delta t$].

Now consider the way things look in the "lab frame". People in the lab frame see the "rest frame" moving with velocity $v\hat{x}$. They will define the force F'_x as

$$F'_x = \frac{dp'_x}{dt'} \, .$$

To work this quantity out in terms of F_x we use the Lorentz transformation: we need $\Delta p'_x$ and $\Delta t'$,

$$\begin{aligned}
\Delta p'_x &= \gamma_v\left(\Delta p_x - \beta_v \Delta E/c\right) \\
&= \gamma_v\left[\Delta p_x - \beta_v (F_x \Delta t)^2/(2mc)\right] \\
\Delta t' &= \gamma_v\left(\Delta t - \beta_v \Delta x/c\right) \\
&= \gamma_v\left[\Delta t - \beta_v F_x \Delta t^2/(2mc)\right] \\
&= \gamma_v \Delta t\left[1 - \beta_v F_x \Delta t/(2mc)\right]
\end{aligned}$$

Now, we divide and take a limit:

$$\begin{aligned}
F'_x &= \lim_{\Delta t' \to 0} \frac{\Delta p'_x}{\Delta t'} \\
&= \lim_{\Delta t \to 0} \frac{\gamma_v \left[\Delta p_x - \beta_v (F_x \Delta t)^2/(2mc)\right]}{\gamma_v \Delta t \left[1 - \beta_v F_x \Delta t/(2mc)\right]} \\
&= \lim_{\Delta t \to 0} \frac{\gamma_v \Delta p_x}{\gamma_v \Delta t} \\
&= F_x \, .
\end{aligned}$$

Components of force parallel to the frames' relative motion are the same in both reference frames!

Suppose the force was in a perpendicular direction. The analysis goes through largely the same:

$$\begin{aligned}
F'_y &= \lim_{\Delta t' \to 0} \frac{\Delta p'_y}{\Delta t'} \\
&= \lim_{\Delta t \to 0} \frac{\Delta p_y}{\gamma_v \Delta t \left[1 - \beta_v F_x \Delta t/(2mc)\right]} \\
&= \lim_{\Delta t \to 0} \frac{\Delta p_y}{\gamma_v \Delta t} \\
&= \frac{F_y}{\gamma_v} \, .
\end{aligned}$$

Components of force perpendicular to the frames' relative motion are related by a factor of $1/\gamma$: The force in the "lab" frame is smaller by $1/\gamma$ than the force in the "rest" frame.

12.4 The force exerted by a current upon a moving charge

Warning: the going gets a little rough in this section.

12.4.1 Lab frame

In the lab frame, we have a wire that is electrically neutral, but that carries a current. The wire contains positive charges with density per unit length $\lambda_+ = \lambda_0$. This is the rest frame of those charges, so this means $\lambda_+^{\mathrm{REST}} = \lambda_0$.

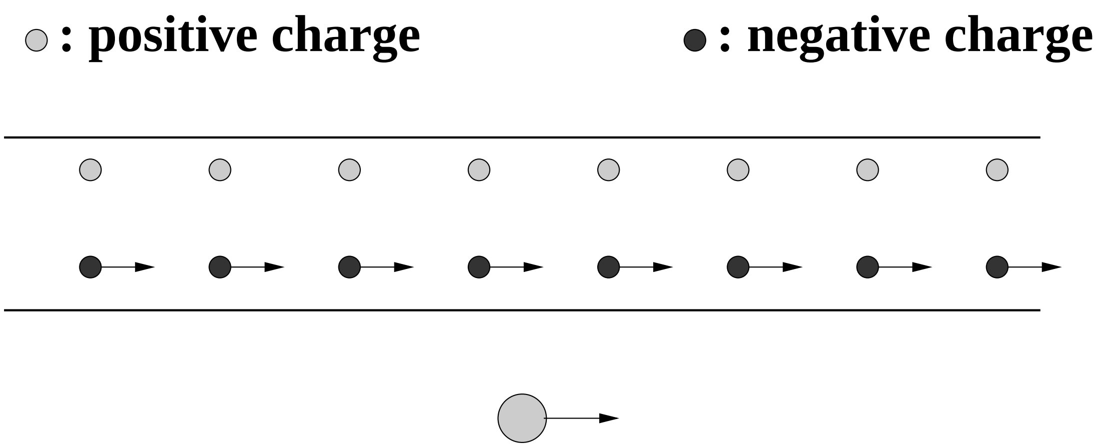


Figure 1: The moving negative charges generate a current moving to the left. This in turn leads to a magnetic field that pushes the charge Q away from the wire.

The wire also contains negative charges; these charges are of course moving with some velocity $\vec{u}$. Since the wire is neutral overall, the density of these negative charges is $\lambda_- = -\lambda_0$. *This is not the density of the electrons in THEIR OWN rest frame!!!!* Suppose that the charge density of the electrons in their own rest frame is $\lambda_-^{\mathrm{REST}}$. By the Lorentz contraction, we must have

$$\lambda_- = \gamma_u \lambda_-^{\mathrm{REST}}$$

where $\gamma_u = 1/\sqrt{1 - u^2/c^2}$. Doing the math, this tells us that

$$\lambda_-^{\mathrm{REST}} = -\frac{\lambda_0}{\gamma_u} \, .$$

We'll need this result shortly.

A charge Q outside of the wire is moving to the right. What forces act on the charge? Since the wire is neutral, there can be no electric force. However, there is a current with magnitude $I = \lambda_0 u$ flowing in the wire. This generates a magnetic field,

$$B = \frac{2I}{rc} = \frac{2\lambda_0 u}{rc}$$

and so there is a force of magnitude

$$F = qvB/c = \frac{2q\lambda_0 uv}{rc^2} \, .$$

By doing the right hand rule game, we quickly see that this force repels the charge from the wire.

12.4.2 Charge's rest frame

By definition, the charge sits still in this frame. Since it's not moving, there *cannot* be a magnetic force that acts on it. But, for there to be a consistent description of the system in the two reference frames there has to be *some kind* of force. What can it be? The only possibility is that there is an *electric* force.

"Where the hell does that come from???" you ask. Let's find out by analyzing the charges in the wire in this frame:

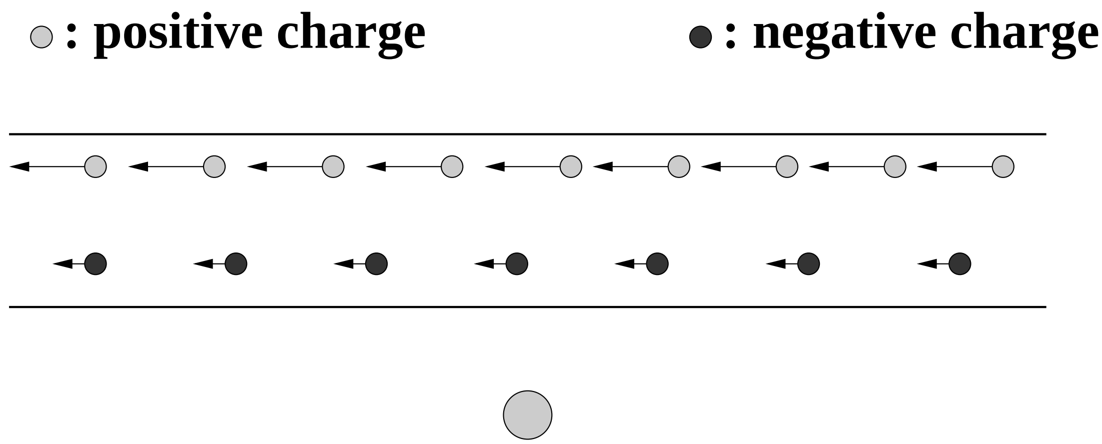


Figure 2: Currents in the wire are irrelevant since the charge does not move in this frame. However, the different length contractions experienced by the positive and the negative charges leads to the wire having a net positive charge as seen in this frame, leading, again, to a force repelling the charge from the wire.

Positive charges: the positive charges are moving in this frame with velocity $-\vec{v}$. Their density as seen in this frame is thus given by

$$\lambda'_+ = \gamma_v \lambda_+^{\mathrm{REST}} = \gamma_v \lambda_0 \ ,$$

where $\gamma_v = 1/\sqrt{1 - v^2/c^2}$.

Negative charges: the negative charges are moving in this frame with a velocity given by adding the velocity $-\vec{u}$ to $\vec{v}$ in the correct, relativistic fashion: their velocity is

$$u' = \frac{u - v}{1 - uv/c^2}$$

and so their density is given by

$$\lambda'_- = \gamma_{u'} \lambda_-^{\mathrm{REST}} = -\frac{\gamma_{u'}}{\gamma_u} \lambda_0 \ .$$

The Lorentz factor $\gamma_{u'}$ is a bit of a mess: we need $1/\sqrt{1 - (u')^2/c^2}$. Let's look at this methodically. First, to simplify the notation a bit, define

$$\begin{aligned} \beta_v &= v/c \\ \beta_u &= u/c \\ \beta_{u'} &= u'/c \ . \end{aligned}$$

Next, look at $1-(u'/c)^2 = 1-\beta_{u'}^2$:

$$\begin{aligned}
1-\beta_{u'}^2 &= 1-\left(\frac{\beta_u-\beta_v}{1-\beta_u\beta_v}\right)^2 \\
&= 1-\frac{\beta_u^2+\beta_v^2-2\beta_u\beta_v}{(1-\beta_u\beta_v)^2} \\
&= \frac{1-2\beta_u\beta_v+\beta_u^2\beta_v^2}{(1-\beta_u\beta_v)^2}-\frac{\beta_u^2+\beta_v^2-2\beta_u\beta_v}{(1-\beta_u\beta_v)^2} \\
&= \frac{1-\beta_u^2-\beta_v^2+\beta_u^2\beta_v^2}{(1-\beta_u\beta_v)^2} \\
&= \frac{(1-\beta_u^2)(1-\beta_v^2)}{(1-\beta_u\beta_v)^2} \\
&= \frac{1}{\gamma_u^2\gamma_v^2(1-\beta_u\beta_v)^2} \ .
\end{aligned}$$

This means that

$$\gamma_{u'} = \gamma_u\gamma_v(1-\beta_u\beta_v) \ .$$

We now finally have enough information to calculate the density of negative charges:

$$\begin{aligned}
\lambda'_- &= -\frac{\gamma_{u'}}{\gamma_u}\lambda_0 \\
&= -\gamma_v(1-\beta_u\beta_v)\lambda_0 \ .
\end{aligned}$$

The *net, total* charge density of the wire in this frame is given by adding the contributions of the negative and the positive charges:

$$\begin{aligned}
\lambda'_{\rm net} &= \lambda'_+ + \lambda'_- \\
&= \gamma_v\lambda_0 - \gamma_v(1-\beta_u\beta_v)\lambda_0 \\
&= \gamma_v\beta_u\beta_v\lambda_0 \ . \\
&= \gamma_v\frac{uv\lambda_0}{c^2} \ .
\end{aligned}$$

The wire has a net positive line charge density in this reference frame!!! This means it generates an electric field

$$\begin{aligned}
E' &= \frac{2\lambda'_{\rm net}}{r} \\
&= \frac{2\gamma_v uv\lambda_0}{rc^2}
\end{aligned}$$

which in turn means that there is a force

$$F' = \frac{2\gamma_v q\lambda_0 uv}{rc^2} \ .$$

repelling it from the wire.

Note that there is also a magnetic field in this frame — there are currents associated with both the positive and the negative charges in the wire. However, since the charge Q is at rest, only the electric field is relevant to the force that it experiences.

12.4.3 Comparison of the forces in the two frames

In the "lab frame", we see the charge moving with speed v and deduce that it will feel a magnetic force

$$F = \frac{2uv\lambda_0}{rc^2} \; .$$

In the "charge frame", the charge is at rest, so it cannot feel any magnetic force. However, we just showed that in this frame the wire is not electrically neutral, and so the charge experiences an electric force of

$$F' = \gamma_v \frac{2uv\lambda_0}{rc^2} \; .$$

Are these results consistent? *Yes!!* Remember that rule for transforming forces: the force that is experienced in the "lab frame" is related to that in the moving frame by a factor of $1/\gamma_v$. In other words, we expect to find

$$F = \frac{1}{\gamma_v} F'$$

— *exactly* what we do in fact find.

You may object that one force is electric and the other is magnetic — aren't we comparing apples and oranges here? For many years, people thought this way: electric forces were one phenomenon, magnetic forces were another. There was no connection between the two. As thought experiments like this show, however, this distinction is a false one. The electric force acting on a charge in that charge's rest frame is *exactly* what we need to explain the magnetic force in a frame in which that charge is moving.

Physics is consistent: even though we give different detailed explanations ascribing what mechanism produces the forces in the two reference frames, we agree *exactly* as to what this force should be. This is the essence of special relativity! It also tells us that electric forces and magnetic forces are really the same thing. "Electricity" and "magnetism" are not separate phenomena: they are different specific manifestations of a single critter, "electromagnetism".

12.5 Summary

The discussion of the past two lectures is probably the most difficult that we will face all semester. Let's recap the major points:

- **Moving clocks run slow.** Time dilation means that a clock which is moving very quickly relative to you will tick more slowly than a clock that is at rest with respect to you.

- **Moving rulers are shortened.** The Lorentz contraction means that you will measure a car shooting past you at very high speed to be shorter than an identical car that is at rest with respect to you.

  - **But only along the direction of motion!!!** There is no Lorentz contraction in a direction orthogonal to the relative motion.

- **What looks like a pure magnetic field in one frame looks like a mixture of magnetic and electric fields in another frame.** This guarantees that there is some kind of force acting on our charge even when we go into its rest frame.