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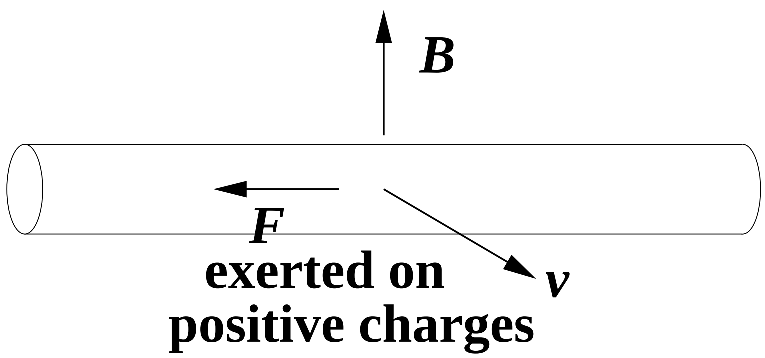
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF PHYSICS
8.022 SPRING 2005

LECTURE 14:
INDUCTION:
FARADAY'S & LENZ'S LAWS

14.1 Moving wires in magnetic fields

14.1.1 Single length

Suppose we take a length of conducting wire and drag it through a uniform magnetic field:



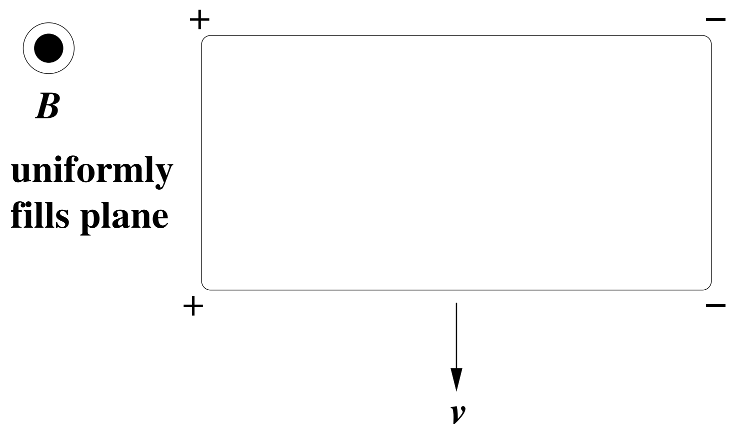
Charges in the wire will feel a force due to being dragged along through the $\vec{B}$-field. This leads to a separation of charges in this rod: as drawn, the left end of the rod will acquire a net + charge, the right end acquires a net − charge.

How much charge separation occurs? By moving the charges apart, we must create an electric field. The electric field will grow until the force that it exerts on the charges balances the magnetic force:

$$q\vec{E}_{\rm balance} + q\frac{\vec{v}}{c} \times \vec{B} = 0$$
$$\longrightarrow E_{\rm balance} = -\frac{\vec{v}}{c} \times \vec{B} \ .$$

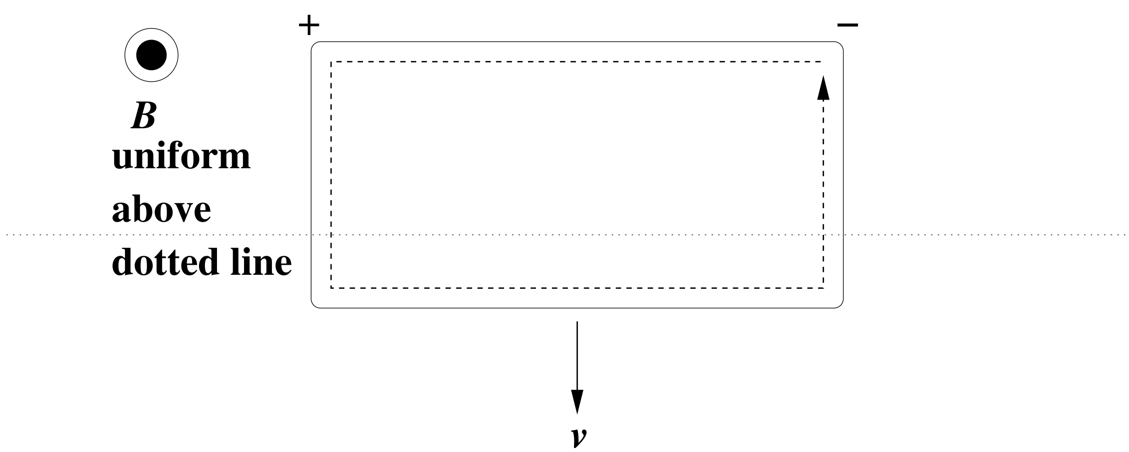
14.1.2 Closed loop

Suppose we now take our wire and bend it into a square. What happens when we drag it through the magnetic field? Well, if the magnetic field is truly uniform and filling all of space, nothing very interesting happens:



We get charge separation in both the top and the bottom legs, so there's an $\vec{E}$-field pointing to the right along both legs. This stops the flow of charge — once the $\vec{E}$-field is built up, the charge separation doesn't do anything else. It's done.

Things get more exciting if we make the magnetic field non-uniform. Suppose that the magnetic field is zero below some line, but is constant and non-zero above that line:



Because of the magnetic force, charge will flow — as we already know — to the left across the upper side of the loop. However, there's now no opposing $\vec{E}$-field that will prevent it from *continuing* to flow! The charges will very happily circumnavigate the entire loop, forming a cyclical current. We say that we have *induced* a current to flow in this loop. This is our first encounter with the phenomenon of electromagnetic induction.

A few other things are worth noticing. First, note that

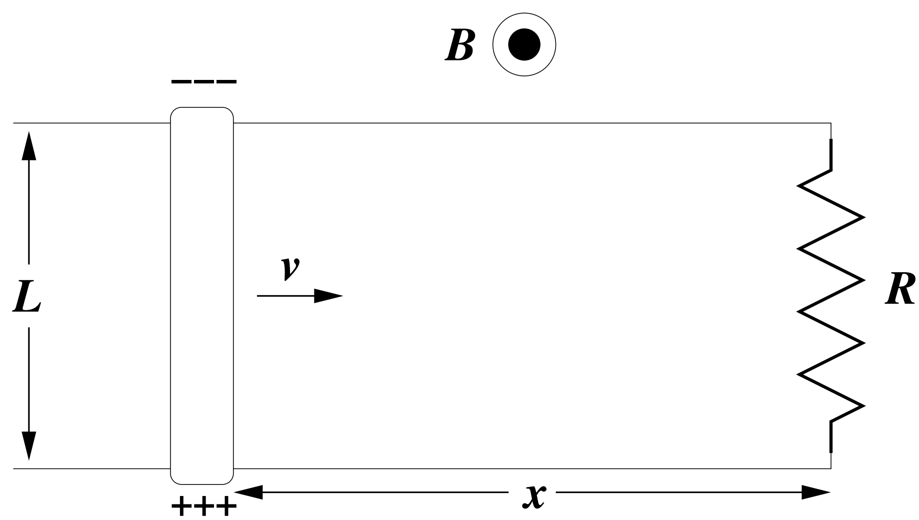
$$\oint_{\text{wire loop}} \vec{E} \cdot d\vec{s} \neq 0 \ .$$

For this to be true, we *cannot* have $\vec{\nabla} \times \vec{E} = 0$! Don't be too disturbed by this — all this tells us is that we have begun to move beyond electrostatics.

Second, notice that the current flowing in this loop will itself feel a force from the magnetic field: the leg across the top of the loop carries a current to the left. Doing the cross product for $\vec{I} \times \vec{B}$, we find that this generates a force on the loop that points up — *opposing* the motion of the loop. (There is also a force to the left on the left-hand side of the loop, and a force to the right on the right-hand side, but these are equal and opposite. The force on the top of the loop is unopposed.) This opposition of the loop's motion is our first contact with a more general rule called *Lenz's law*, which is a kind of electromagnetic inertia.

14.2 Induced EMF

This little example may give you an idea for how to make a simple little generator for electric power. Imagine sliding a bar across a pair of rails, sitting in a region of constant magnetic field:



The motion of the bar causes charge separation, which in turn will drive a current to flow around the loop and through the resistor. This is the same behavior we would have seen if the current were driven by a battery rather than by the moving rod! We can thus think of the motion of the rod as acting like a source of EMF.

What is the value of this EMF? For a battery, we defined the EMF as the potential of the positive terminal with respect to the negative terminal. Bearing in mind that potential is the work (per unit charge) it takes to move charges from point to point, it is simple to compute the EMF of this arrangement:

$$\begin{aligned} \mathcal{E} &= \frac{1}{q} W(\text{moving from } - \rightarrow +) \\ &= \frac{1}{q} \int_{-}^{+} \vec{F} \cdot d\vec{s} \\ &= \frac{1}{c} \int_{0}^{L} \left(\vec{v} \times \vec{B} \right) \cdot d\vec{s} \\ &= \frac{vBL}{c} \,. \end{aligned}$$

The current that flows through this loop is then given by

$$I_{\rm ind} = \frac{\mathcal{E}}{R} = \frac{vBL}{cR}\ .$$

14.3 A different way to think about this result

The formula we just derived gives us an EMF that is proportional to $v = dx/dt$, the speed of the bar sliding along the rails. The magnetic flux in the rectangle defined by the rails, the bar, and the resistor is given by

$$\Phi_B = Blx\ .$$

Because of the motion of the rod, this flux decreases with time: taking the derivative, we have

$$\frac{\partial \Phi_B}{\partial t} = -Blv\ .$$

This is (almost) exactly what we just got for the EMF generated by the rod's motion:

$$\mathcal{E} = -\frac{1}{c}\frac{\partial \Phi_B}{\partial t}\ .$$

As we'll discuss in a moment, this formula turns out to be the most general formula we need to describe induced EMF.

For now, the most important thing to understand about this formula is the minus sign. The minus tells us about the direction of the induced EMF; or, if you prefer, the direction in which a current driven by this induced EMF tends to flow. The current flows in such a direction that it *opposes the change in flux* through the loop. This is Lenz's law: "I don't want the flux through me to change, dammit[1]". Lenz's law expresses a kind of inertia for magnetic flux.

Let's examine this carefully. Suppose the bar were sitting still. The flux out of the page is some positive number, since $\vec{B}$ points out of the page, and the area is oriented so that its normal is likewise out of the page (by right-hand rule). Now, we make the bar slide to the right. The *outward flux gets smaller* since the area is decreasing in size. Lenz's law tells us that the current from the induced EMF will fight this change: to oppose the decrease in flux, the current flows in such a way as to augment it. This means it must flow counterclockwise: by right hand rule, a clockwise circulating current will tend to increase the magnetic field pointing out of the page.

If the bar slides to the left, the outward flux would *grow*. Lenz's law wants to fight that too, so it induces a current that will tend to decrease the magnetic field out of the page — a current circulating *clockwise*.

If a current flows in a magnetic field, it feels a force. In this example of a sliding bar, Lenz's law guarantees that this force acts to oppose the bar's motion — it ends up acting like a drag force. Consider the bar sliding to the right: the current circulates counter clockwise, so the $\vec{I} \times \vec{B}$ force points to the left, opposing the bar's motion. Consider the bar sliding to the left: the current circulates clockwise, so the $\vec{I} \times \vec{B}$ force points to the right — again,

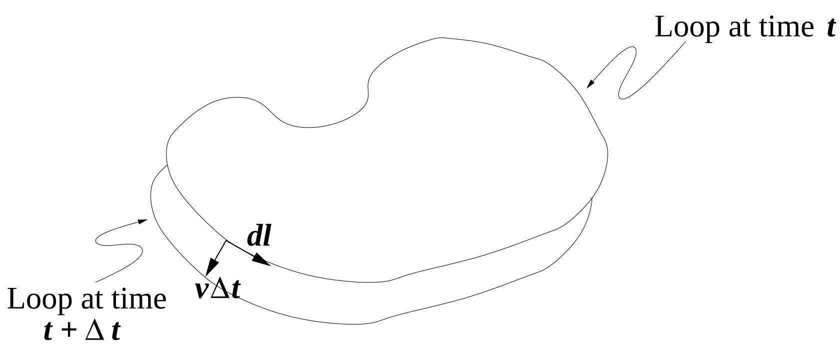
[1]The final "dammit" is in honor of Professor Ray Ashoori, who, during the Spring 2003 semester, was apparently unable to state Lenz's law without a mild expletive.

opposing the bar's motion. In both cases, the interaction of the current with the magnetic field acts as a drag. The minus sign in the EMF equation was *key* to this: if it had been a + sign, we would have found "anti-drag"! Rather than acting to slow down the bar, the magnetic field and magnetic induction would act to *speed it up*!!! You should be able to convince yourself that this is absurd: you quickly end up with a "runaway" configuration, where the bar's speed just increases without limit. This would end up violating conservation of energy. Lenz's law and the minus sign can thus be regarded as enforcing conservation of energy in magnetic induction.

14.4 EMF and $\partial\Phi_B/\partial t$: General proof for static fields

We showed for a specific example that the EMF generated by moving things around in a magnetic field is given by $-(1/c)\partial\Phi_B/\partial t$. We now prove that this result in fact holds in general.

To do so, consider a loop of arbitrary shape. Take this loop to be moving with velocity $\vec{v}$ through an arbitrary — but static — magnetic field $\vec{B}$:



At time t, the magnetic flux through this loop is

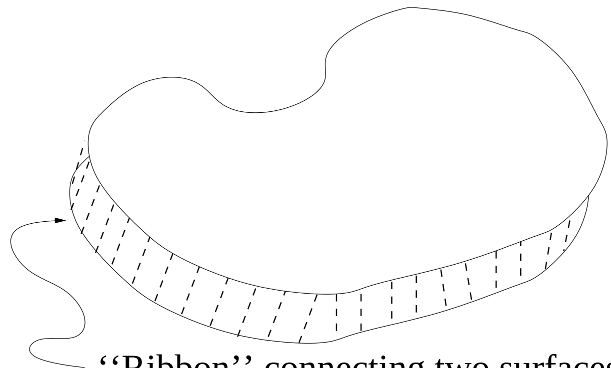
$$\Phi_B(t) = \int_S \vec{B} \cdot d\vec{a}$$

where S is any surface spanned by the wire at time t.

How does the flux change when it moves to its new position at $t + \Delta t$? By looking at the picture, we can see that

$$\Delta\Phi_B = \Phi_B(t + \Delta t) - \Phi_B(t)$$

is just given by the flux through a "ribbon" connecting the two surfaces:



In other words,

$$\Delta\Phi_B = \int_{\text{ribbon}} \vec{B} \cdot d\vec{a}\,.$$

Now, we want to think about how we actually do this integral. Before setting it up, let's think for a moment about charges that might be moving in the wire loop. Let's suppose a charge's velocity along the loop is $\vec{u}$. The *total* velocity of the change is then $\vec{v}_{\text{charge}} = \vec{u} + \vec{v}$ — we add the velocity along the loop to the "external" velocity of the loop.

OK, now think about the area element on the ribbon. By staring at the figures a little bit and thinking about what the area element means, we can see that

$$d\vec{a} = (\vec{v}\,\Delta t) \times d\vec{l}\,.$$

In fact, since $\vec{u}$ is parallel to $d\vec{l}$, we can just as well write this as

$$d\vec{a} = (\vec{v}_{\text{charge}}\Delta t) \times d\vec{l}$$

— this is *exactly* the same as the previous way of writing $d\vec{a}$ since $\vec{u} \times d\vec{l} = 0$.

We have thus reduced our *area* integral to a *line* integral: we just have to integrate with respect to $d\vec{l}$ around the wire loop. In other words,

$$\Delta\Phi_B = \oint_{\text{loop}} \vec{B} \cdot \left[(\vec{v}_{\text{charge}}\Delta t) \times d\vec{l}\right]\,.$$

Since we are holding Δt fixed while we do the integral, we can divide it out and take the limit $\Delta t \to 0$:

$$\frac{\partial\Phi_B}{\partial t} = \oint_{\text{loop}} \vec{B} \cdot \left(\vec{v}_{\text{charge}} \times d\vec{l}\right)\,.$$

Almost there! To massage this further, note that, for any three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$,

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}\,.$$

(This is very simple to prove. The most straightforward way is to just write out $\vec{a} = a_x\hat{x} + a_y\hat{y} + a_z\hat{z}$, etc, and grind. If you know about the properties of matrix determinants, you can probably come up with an even quicker proof.) We then rearrange our integrand to give us

$$\begin{aligned}\frac{\partial \Phi_B}{\partial t} &= \oint_{\text{loop}} \left(\vec{B} \times \vec{v}_{\text{charge}}\right) \cdot d\vec{l} \\ &= -\oint_{\text{loop}} \left(\vec{v}_{\text{charge}} \times \vec{B}\right) \cdot d\vec{l} \\ &= -c\oint_{\text{loop}} \left(\frac{\vec{v}_{\text{charge}}}{c} \times \vec{B}\right) \cdot d\vec{l}\end{aligned}$$

The quantity in parentheses on the last line is just the magnetic force (per unit charge). By integrating it over a closed loop, we are computing the work done, per unit charge, to take it around the loop. By definition, this is the EMF! We thus have finally proven our general result:

$$\frac{\partial \Phi_B}{\partial t} = -c\mathcal{E}$$

or

$$\mathcal{E} = -\frac{1}{c}\frac{\partial \Phi_B}{\partial t} .$$

14.5 A few subtleties

14.5.1 Work

In the above discussion, I claimed that $(\vec{v}_{\text{charge}}/c) \times \vec{B}$, integrated over a closed loop, is the work done per unit charge to go around the loop; this is how I defined the EMF $\mathcal{E}$ we just calculated. This hopefully bothered the hell out of you: a few lectures ago, I claimed that magnetic forces never do work on charges!

Am I liar? Not in this instance; instead, there is something subtle going on. Work is being done — but it is not actually being done by the magnetic field! Instead, the work is being done by whatever "external agent" moves the loop. The loop could be moving because it is being dragged by a team of sled dogs; or, it could be part of an electric generator; or it could be falling in a gravitational field. In cases like this, one can look more carefully at the work that is being done by this "agent" and show that it is precisely what is needed to get the EMF I claimed above.

14.5.2 Holding loop fixed, moving the field

In the above discussion, we have fixed our magnetic field and moved our loop of wire. What if instead we held our loop fixed and moved the magnetic field? (Or rather, the source of the field?)

From our discussion of special relativity, you should be able to guess the answer: we had best get the same result! Moving the magnetic field and moving the loop have to be exactly equivalent to one another — we're just jumping from one frame to another.

However, when the loop is held steady, the charges don't move. There is thus no way for a magnetic field to exert a force on them. The only way to reconcile the current that we see induced in the loop is to conclude that there must be an electric field present.

This electric field *cannot* be a "nice" electric field, like we got to know back when we only worried about electrostatics. It is an electric field that drives current to run around in a circle. From the idea that the EMF is the work per unit charge, it follows that

$$\mathcal{E} = \oint_C \vec{E}_{\rm ind} \cdot d\vec{l}$$

where $\vec{E}_{\rm ind}$ is the electric field that is *induced* by the changing magnetic field, and C is the closed path around the loop. But we also have

$$\mathcal{E} = -\frac{1}{c}\frac{\partial \Phi_B}{\partial t} \; .$$

Let's put these two equations together:

$$\begin{aligned}\oint_C \vec{E} \cdot d\vec{l} &= -\frac{1}{c}\frac{\partial \Phi_B}{\partial t} \\ &= -\frac{1}{c}\frac{\partial}{\partial t}\int_S \vec{B} \cdot d\vec{a} \; .\end{aligned}$$

The surface S is any surface which has the closed path C as its boundary. (We drop the subscript "ind" on the electric field from now on; it should be clear that we are discussing an electric field that arises from the variations of the magnetic field.)

The logic should smell kind of familiar at this point. We've got a line integral on the left, and a surface integral on the right. To put things into a "nice" form, we want to change our line integral into a surface integral. We thus invoke Stokes' theorem:

$$\oint_C \vec{E} \cdot d\vec{l} = \int_S (\vec{\nabla} \times \vec{E}) \cdot d\vec{a} \; .$$

Plugging this in, we find

$$\begin{aligned}\int_S \left(\vec{\nabla} \times \vec{E}\right) \cdot d\vec{a} &= -\frac{1}{c}\frac{\partial}{\partial t}\int_S \vec{B} \cdot d\vec{a} \\ \int_S \left(\vec{\nabla} \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t}\right) \cdot d\vec{a} &= 0\end{aligned}$$

Since this must hold for an arbitrary surface S bounded by the curve C, we conclude that

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \; .$$

This equation quantitatively relates the varying magnetic field to the induced electric field.

14.6 Faraday's law

The most important equations that we have derived in this lecture are all different variants of what is called **Faraday's law**:

$$\mathcal{E} = \oint_C \vec{E} \cdot d\vec{l} = -\frac{1}{c}\frac{\partial \Phi_B}{\partial t} = -\frac{1}{c}\frac{\partial}{\partial t}\int_S \vec{B} \cdot d\vec{a}$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial B}{\partial t} \ .$$

The first version is the "integral form" of Faraday's law; the second is the "differential form".

It's worth stopping at this point and gathering together all of the equations we've got so far describing divergences and curls of electric and magnetic fields:

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{J} \ .$$

These are *almost* the Maxwell's equations that completely describe the electromagnetic field. There's just one tiny little piece missing. Maxwell figured out what this missing piece was just by arguing that the two curl equations should exhibit a kind of symmetry which is lacking in their current form. You may be able to figure out what's missing by taking the divergence of the last equation and noticing that there's an inconsistency. Once you fix up this inconsistency, you get a set of equations suitable for printing on a t-shirt.