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LECTURE 15:
MUTUAL AND SELF INDUCTANCE.

15.1 Using induction

Induction is a fantastic way to create EMF; indeed, almost all electric power generation in the world takes advantage of Faraday's law to produce EMF and drive currents. Given the defining formula,

$$\mathcal{E} = -\frac{1}{c}\frac{d\Phi_B}{dt}$$

where

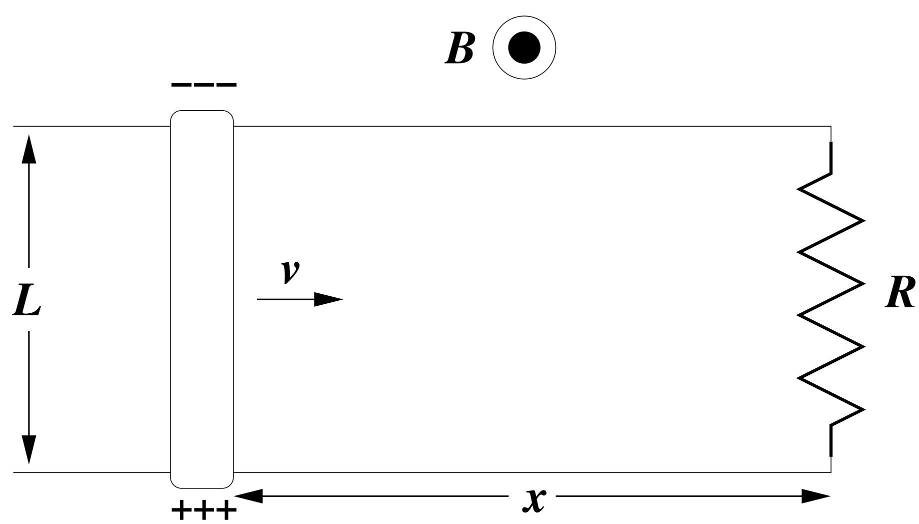
$$\Phi_B = \int_S \vec{B} \cdot d\vec{A}$$

all that we need to do is set up some way to make a time varying magnetic flux and we can induce EMF and drive currents very happily.

Given the definition of magnetic flux, there are essentially three ways that we can make it vary and thereby create an EMF: we can make the area vary; we can make the "dot product" vary; and we can make the magnetic field vary. Let's look at these three situations one by one.

15.1.1 Changing area

We already discussed exactly this situation when we looked at the rod sliding on rails in a uniform magnetic field:

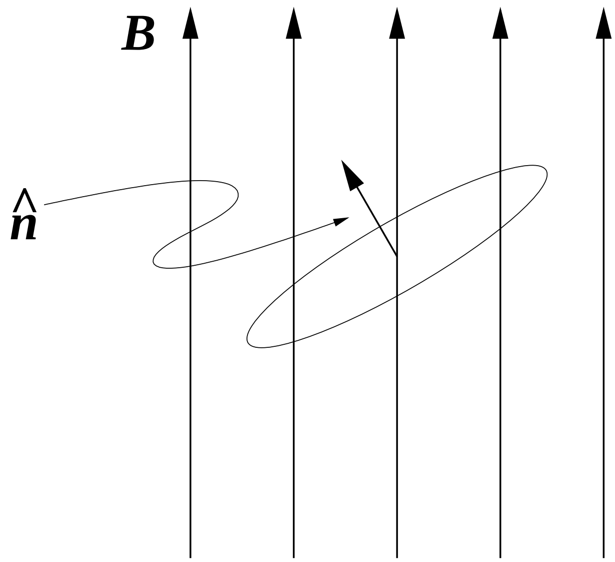


The changing flux in this case is due to the continually changing area. We get an EMF whose magnitude is $\mathcal{E} = BLv/c$, and that (by Lenz's law) drives a current which circulates counterclockwise in the loop.

In principle, we could imagine using this as a power source. For example, we could replace the resistor with a lightbulb and use this arrangement as a lamp. However, you'd have to find some way to continually slide the bar: the energy radiated by the lamp has to come from somewhere! In this case, it would come from the poor schmuck who continually drags the bar back and forth.

15.1.2 Changing "dot product"

Another way to generate an EMF is to make the "dot product" — i.e., the relative orientation of our area and the magnetic field — vary with time. Imagine we have a circular loop sitting in a uniform magnetic field:



The magnetic flux through the loop is given by

$$\begin{aligned}\Phi_B &= \pi r^2 \vec{B}\cdot\hat{n}\\ &= \pi r^2 B\cos\theta\end{aligned}$$

where θ is the angle between the loop's normal vector and the magnetic field. Now, suppose we make the loop rotate: say the angle varies with time, $\theta = \omega t$. The EMF that is generated in this case is then given by

$$\mathcal{E} = \frac{\omega\pi r^2 B}{c}\sin\omega t\ .$$

This kind of arrangement is in fact exactly how most power generators operate: An external force (flowing water, steam, gasoline powered engine) forces loops to spin in some magnetic field, generating electricity. Note that the EMF that is thereby generated is sinusoidal — generators typically produce AC (alternating current) power.

15.1.3 Changing field

Finally, in many situations we can make the magnetic field itself vary. Suppose the integral is very simple, so that it just boils down to field times area. If the magnetic field is a function of time, then the flux will be a function of time:

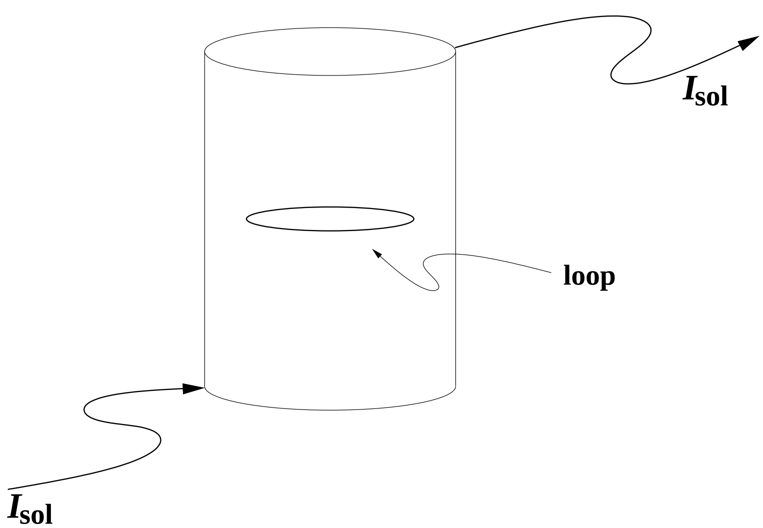
$$\Phi_B(t) = B(t)A \; .$$

We just need to take a derivative, divide by c, and we have an EMF.

This method of creating an EMF is one of the most important. In practice, it typically comes about when our magnetic field comes from some current that varies in time. This is the case that we will focus on in the remainder of this lecture.

15.2 Inductance example

Consider the following arrangement: we place a loop of wire with radius r in the center of a long solenoid. The solenoid has a wire that is wrapped N times around its total length l; hence, the number of turns per unit length is N/l. A current $I_{\rm sol}$ runs through this solenoid; this current varies with time. What is the EMF $\mathcal{E}_{\rm loop}$ that is induced in the wire loop?



We begin with our defining relation,

$$\mathcal{E} = \frac{1}{c}\frac{d\Phi_B}{dt} \; .$$

(In what follows, I will often drop the minus sign. This is fine provided we bear in mind that we need to use Lenz's law to tell us the direction in which any induced current will flow.) The magnetic flux is given by the magnetic field of the solenoid times the cross sectional area of the loop:

$$\begin{aligned} \Phi_B &= B_{\rm sol}A_{\rm loop} \\ &= \frac{4\pi N I_{\rm sol}}{cl}\pi r^2 \; . \end{aligned}$$

(Recall that the current per unit length is nI if the input current is I and the number of turns per unit length is n.) This means that the EMF induced in the little loop is

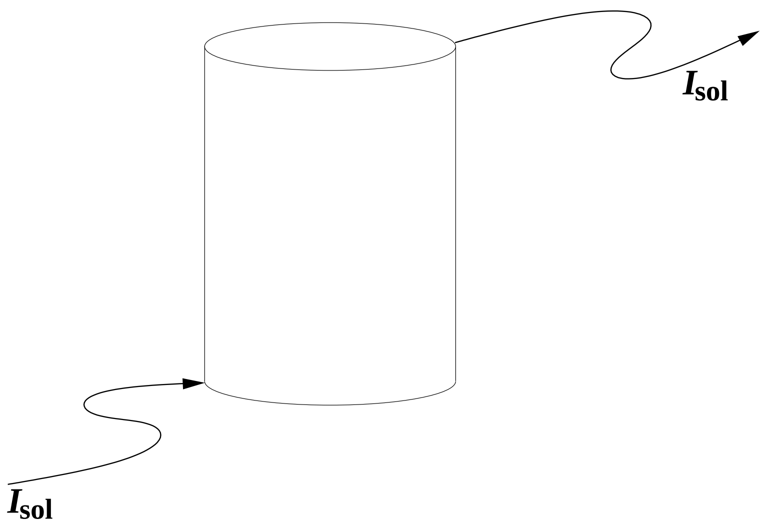
$$\mathcal{E}_{\rm loop} = \frac{4\pi^2 r^2 N}{c^2 l}\frac{dI_{\rm sol}}{dt}\ .$$

In words, the EMF induced in the loop is totally set by the rate of change of current in the solenoid, times a bunch of constants that depend on the geometry of our setup.

We will return to look at this example somewhat more systematically.

15.3 Self inductance

Suppose we take the loop out of the solenoid:



Is there any EMF induced in this case? Let's think about this carefully: The magnetic field threading the solenoid is

$$B = \frac{4\pi N I_{\rm sol}}{cl}\ .$$

The magnetic flux through *each loop* that constitutes the solenoid is

$$\Phi_B^{1\ \rm loop} = \frac{4\pi^2 R^2 N I_{\rm sol}}{cl}$$

where R is the solenoid's radius. This means that the *total* flux as far as any induction is concerned must be $\Phi_B^{\rm tot} = N\Phi_B^{1\ \rm loop}$ — we get a contribution of $\Phi_B^{1\ \rm loop}$ from every loop that constitutes the solenoid! In other words, we end up with

$$\Phi_B^{\rm tot} = \frac{4\pi^2 R^2 N^2 I_{\rm sol}}{cl}\ .$$

Taking the time derivative, we find the EMF that is induced when this current varies:

$$\mathcal{E} = \frac{4\pi^2 R^2 N^2}{c^2 l}\frac{dI_{\rm sol}}{dt}\ .$$

Again, we have a bunch of stuff that just depends on the solenoid's geometry (and constants) times the rate of change of the current.

The math hopefully makes sense. What you should be asking yourself at this point is "OK, fine, I've got an EMF, but what *IS* it??" Lenz's law helps us to understand what this EMF is and does: Suppose the current is increasing, so that the magnetic field is likewise increasing. Lenz's law tells us that the induced EMF will try to "fight" this: the EMF that is induced will tend to drive currents which *oppose* the change in magnetic flux. For this setup, this means that the EMF will oppose the increase — it will drive a current which tries to prevent B from changing. If the current was decreasing, the EMF will oppose the decrease. The punchline is that *inductance opposes the change in the current*. The EMF that is induced in a situation like this is often called the "back EMF" since it acts "back" on the circuit to try to hold the current steady.

From this situation, we define a quantity called the "self-inductance", L:

$$\mathcal{E} = L\frac{dI}{dt} .$$

Quantities which have a self-inductance and are used in circuits (as we'll study next lecture) are called "inductors". For a solenoidal inductor, we can read off the value of L from the preceding calculation:

$$L_{\rm sol} = \frac{4\pi^2 N^2 R^2}{c^2 l} .$$

Bear in mind that this formula *only* holds for solenoids! The solenoid is, however, a very important example of an inductor and crops up quite a lot.

15.3.1 Units

Since EMF has units of potential, and dI/dt is [charge]/[time]^2, it is simple to deduce the units of self-inductance. In cgs units, we have

$$\frac{[\text{Potential}]}{[\text{charge}]/[\text{time}]^2} \to \frac{\text{esu/cm}}{\text{esu/sec}^2} \to \frac{\text{sec}^2}{\text{cm}} .$$

Inductance is measured in the rather odd unit of sec^2/cm in cgs units.

In SI units, we have

$$\frac{[\text{Potential}]}{[\text{charge}]/[\text{time}]^2} \to \frac{\text{Volts}}{\text{Amp/sec}} .$$

We could reduce this further, but what is done instead is to give this combination a name: 1 Volt/(Amp sec^{-1}) = 1 Henry. This unit is named after Joseph Henry, who discovered self-inductance. There's no question it's a pretty goofy name for a unit, though.

15.4 Energy stored in an inductor

Suppose we have an inductor that is sitting on its own, and we somehow force a current to flow through it. As soon as the current level starts changing, a back EMF is induced that opposes the flow of the current. We can *force* the current to flow by fighting this back EMF;

but, it takes power to do so. To overcome the back EMF of the inductor, we must supply a power to the inductor of

$$P = I\mathcal{E} = LI\,\frac{dI}{dt}\ .$$

Suppose we increase the current flowing through the inductor from $I = 0$ at $t = 0$ to I at some later time t. How much work is done to do this? Since $P = dW/dt$, we just integrate:

$$\begin{aligned} W &= \int_0^t P\,dt \\ &= \int_0^t LI\,\frac{dI}{dt}\,dt \\ &= \int_0^I LI\,dI \\ &= \frac{1}{2}LI^2\ . \end{aligned}$$

Since we did this much work to get the current flowing, this is the energy that is stored in the inductor:

$$U_L = \frac{1}{2}LI^2\ .$$

15.4.1 Magnetic field energy

How exactly is this energy stored in the inductor? The key thing that we changed is that we created a magnetic field where there was none previously. This energy must essentially be the energy that is stored in the magnetic field we have just created! (Since we previously talked about energy stored in electric fields, and since electric and magnetic fields are essentially the same thing from the viewpoint of special relativity, it shouldn't be too much of a surprise that we can discuss the energy of magnetic fields as well.)

It is easy to see that a magnetic field has an energy density by looking at the specific example of the solenoidal inductor. Suppose a current I is flowing in the solenoid discussed previously. The energy that is stored in it is $U_L = \frac{1}{2}LI^2$. We also know that $L = 4\pi^2N^2R^2/(c^2l)$. Let's write this out and reorganize things:

$$\begin{aligned} U_L &= \frac{1}{2}LI^2 \\ &= \frac{2\pi^2N^2R^2}{c^2l}I^2\ . \end{aligned}$$

The volume of the solenoid is just $V = \pi R^2 l$ (it's a cylinder of radius R and height l). The magnetic field of the solenoid is $B = 4\pi NI/(cl)$. Knowing this, we can rewrite the above formula as

$$\begin{aligned} U_L &= \frac{\pi R^2 l}{8\pi}\left(\frac{4\pi NI}{cl}\right)^2 \\ &= (\text{Volume})\,\frac{B^2}{8\pi}\ . \end{aligned}$$

This means that the magnetic field has an energy per unit volume of $B^2/8\pi$ — a result that is *VERY* similar to the energy per unit volume of the electric field!

This is one of the nice things about cgs units: it really brings out the symmetry between the fields. If u_X is the energy per unit volume of the X field, then we have

$$\begin{aligned} u_E &= \frac{E^2}{8\pi} \\ u_B &= \frac{B^2}{8\pi} . \end{aligned}$$

In SI units, it's not quite so symmetric:

$$\begin{aligned} u_E &= \frac{\epsilon_0 \, E^2}{2} \\ u_B &= \frac{B^2}{2\mu_0} . \end{aligned}$$

15.4.2 Using energy to find L

In many cases, the result that the energy density $u_B = B^2/8\pi$ and the total energy $U = \frac{1}{2}LI^2$ can be used as a tool for computing the self inductance. This is sometimes easier than looking at a time varying current, calculating the EMF induced, and reading out the inductance L. Here's a simple recipe that often works very nicely:

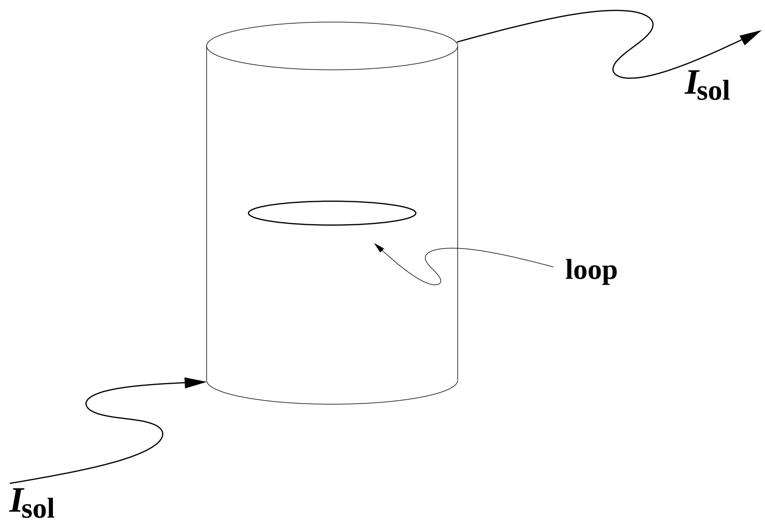
- Compute the magnetic field.
- Integrate $B^2/8\pi$ over a volume.
- Divide by I^2; multiply by 2.

What is left must be the inductance L.

You will have an opportunity to test drive this trick on a pset.

15.5 Mutual inductance and reciprocity

Let's return to the first example that we discussed: the loop inside the solenoid. Let us label the solenoid with a "1" (i.e., it is "circuit element 1"), and the loop with a "2". We



found that the EMF induced in the loop was equal to some factor times the rate of change of current in the solenoid. Using our new labeling system, we would say

$$\mathcal{E}_2 = M_{21}\frac{dI_1}{dt}\ .$$

The coefficient M_{21} is called the *mutual inductance*; in this case it is the mutual inductance from element 1 to element 2. For this setup, it is something we already worked out:

$$M_{21} = \frac{4\pi^2 r^2 N}{c^2 l}$$

where r is the radius of the small loop.

Suppose that we now reverse things: rather than putting our time varying current through the solenoid, we put it through the small loop. By definition, the EMF induced in the solenoid by this time changing current must be described by

$$\mathcal{E}_1 = M_{12}\frac{dI_2}{dt}\ .$$

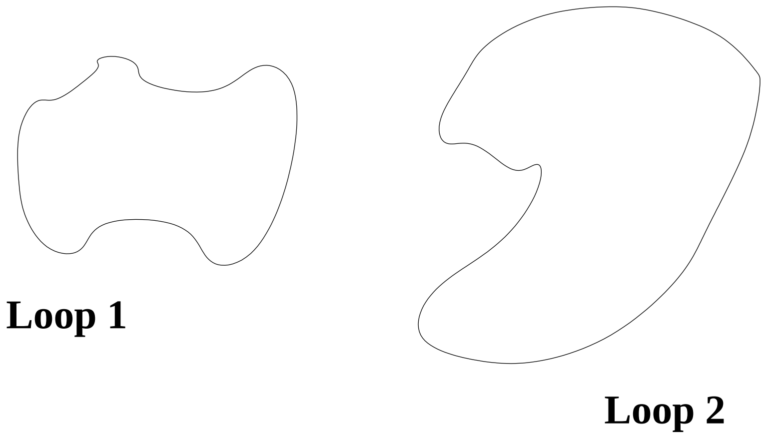
However, *how* do we calculate M_{12}? It should be a total mess — the B field from the loop is *highly* non trivial, and the geometry of the solenoid through which we need to calculate the flux is very complicated. It seems like there is *no hope* to possibly calculate this!

It turns out there is an amazing theorem — the "reciprocity theorem" — that makes it all easy. The reciprocity theorem tells us that

$$M_{12} = M_{21}\ .$$

Operationally, this means we can solve the "hard" problem — induced EMF in solenoid due to changing currents in loop — by first solving the "easy" problem — induced EMF in loop due to changing currents in solenoid. This is fantastic news for solving nasty problems.

The proof of this result is a very nice application of the vector potential. Consider two loops of wire,



Suppose that a current I runs through loop 1. What is the flux of magnetic field through loop 2 due to the current in loop 1? By definition, this flux will be

$$\Phi_{21} = \int_{S_2} \vec{B}_1 \cdot d\vec{a}_2$$

where $\vec{B}_1$ is the magnetic field arising from loop 1, S_2 is the surface spanned by loop 2, and $d\vec{a}_2$ is an area element on this surface. (You'll see why I choose a lower case a for the area element in just a moment.) Now, we rewrite this result in terms of the vector potential arising from loop 1:

$$\begin{aligned} \Phi_{21} &= \int_{S_2} \left(\vec{\nabla} \times \vec{A}_1\right) \cdot d\vec{a}_2 \\ &= \oint_{C_2} \vec{A}_1 \cdot d\vec{l}_2 \ . \end{aligned}$$

On the last line, we've used Stokes' theorem to reexpress the flux as a line integral of the vector potential. Finally, we use the fact that the vector potential from a current in loop 1 to some field point is

$$\vec{A}_1 = \frac{I}{c} \oint_{C_1} \frac{d\vec{l}_1}{r} \ .$$

With this, the flux becomes

$$\Phi_{21} = \frac{I}{c} \oint_{C_1} \oint_{C_2} \frac{d\vec{l}_1 \cdot d\vec{l}_2}{r_{21}}$$

where r_{21} is the distance from the length element $d\vec{l}_1$ to the length element $d\vec{l}_2$.

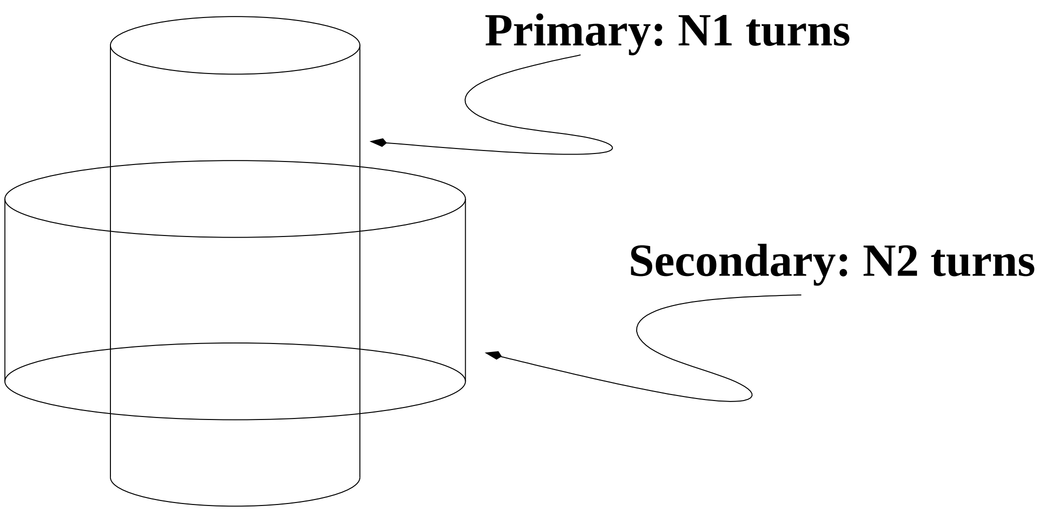
Now, what happens if I put the current I in loop 2 and ask what is the flux that goes through loop 1? Going through all of these steps again, we find

$$\Phi_{12} = \frac{I}{c} \oint_{C_2} \oint_{C_1} \frac{d\vec{l}_2 \cdot d\vec{l}_1}{r_{12}} \ .$$

This is the exact same thing as Φ_{21}! Since the fluxes are the same, the derivative of the fluxes must be the same; it follows that the mutual inductance $M_{12} = M_{21}$, and the reciprocity theorem is proved.

15.6 Transformers

A very important *practical* application of mutual inductance is in the construction of transformers: devices which step up (or step down) an AC voltage. The simplest transformer consists of a pair of nested solenoids:



The idea is that we drive a time varying current through the primary, making a time varying magnetic flux:

$$\mathcal{E}_1 = \frac{N_1}{c}\frac{d\Phi_B}{dt}$$

where Φ_B is the magnetic flux from a single turn. Since the same flux threads the second solenoid, the induced EMF in the outer solenoid is

$$\mathcal{E}_2 = \frac{N_2}{c}\frac{d\Phi_B}{dt} .$$

Comparing, we have

$$\mathcal{E}_2 = \mathcal{E}_1\frac{N_2}{N_1} .$$

By changing the ratio of turns, we can make enormous voltages ... or, we can step voltages down.