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MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF PHYSICS
8.022 SPRING 2005

LECTURE 19:
DISPLACEMENT CURRENT. MAXWELL'S EQUATIONS.

19.1 Inconsistent equations

Over the course of this semester, we have derived 4 relationships between the electric and magnetic fields on the one hand, and charge and current density on the other. They are:

$$\begin{aligned}
\text{Gauss's law:} \qquad & \vec\nabla\cdot\vec E = 4\pi\rho \\
\text{Magnetic law:} \qquad & \vec\nabla\cdot\vec B = 0 \\
\text{Faraday's law:} \qquad & \vec\nabla\times\vec E = -\frac{1}{c}\frac{\partial\vec B}{\partial t} \\
\text{Ampere's law:} \qquad & \vec\nabla\times\vec B = \frac{4\pi\vec J}{c}
\end{aligned}$$

As written, these equations are slightly inconsistent. We can see this inconsistency very easily — we just take the divergence of both sides of Ampere's law. Look at the left hand side first:

$$\text{Left hand side:} \qquad \vec\nabla\cdot\left(\vec\nabla\times\vec B\right) = 0\ .$$

This follows from the rule that the divergence of the curl is always zero (as you proved on pset 3). Now look at the right hand side:

$$\text{Right hand side:} \qquad \vec\nabla\cdot\left(\frac{4\pi\vec J}{c}\right) = -\frac{4\pi}{c}\frac{\partial\rho}{\partial t}\ .$$

Here, I've used the continuity equation, $\vec\nabla\cdot\vec J = -\partial\rho/\partial t$.

Ampere's law is inconsistent with the continuity equation except when $\partial\rho/\partial t = 0$!!! A charge density that is constant in time is actually a fairly common circumstance in many applications, so it's not too surprising that we can go pretty far with this "incomplete" version of Ampere's law. But, as a matter of principle — and, as we shall soon see, of practice as well — it's just not right. We need to fix it somehow.

19.2 Fixing the inconsistency

We can fix up this annoying little inconsistency by inspired guesswork. When we take the divergence of Ampere's left hand side, we get zero — no uncertainty about this whatsoever, it's just ZERO. We should be able to add a function to the right hand side such that the divergence of the right side is forced to be zero as well.

Let's suppose that our "generalized Ampere's law" takes the form

$$\vec\nabla\times\vec B = \frac{4\pi\vec J}{c} + \vec{\mathcal F}\ .$$

Our goal now is to figure out what $\vec{\mathcal{F}}$ must be. Taking the divergence of both sides, we find

$$\begin{aligned} 0 &= \frac{4\pi\vec{\nabla}\cdot\vec{J}}{c} + \vec{\nabla}\cdot\vec{\mathcal{F}} \\ \longrightarrow \vec{\nabla}\cdot\left(c\vec{\mathcal{F}}\right) &= 4\pi\frac{\partial\rho}{\partial t} . \end{aligned}$$

Our mystery function has the property that when we take its divergence (and multiply by c), we get the rate of change of charge density.

This looks a lot like Gauss's law! If we take the time derivative of Gauss's law, we have

$$\begin{aligned} \frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{E} &= 4\pi\frac{\partial\rho}{\partial t} \\ \longrightarrow 4\pi\frac{\partial\rho}{\partial t} &= \vec{\nabla}\cdot\left(\frac{\partial\vec{E}}{\partial t}\right) \end{aligned}$$

On the last line, I've use the fact that it is OK to exchange the order of partial derivatives:

$$\begin{aligned} \frac{\partial}{\partial t}\vec{\nabla} &= \hat{x}\frac{\partial}{\partial t}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial t}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial t}\frac{\partial}{\partial z} \\ &= \hat{x}\frac{\partial}{\partial x}\frac{\partial}{\partial t} + \hat{y}\frac{\partial}{\partial y}\frac{\partial}{\partial t} + \hat{z}\frac{\partial}{\partial z}\frac{\partial}{\partial t} \\ &= \vec{\nabla}\frac{\partial}{\partial t} . \end{aligned}$$

(This actually works in any coordinate system; Cartesian coordinates are good enough to demonstrate the point.)

Substituting in for $4\pi\,\partial\rho/\partial t$, we have

$$\vec{\nabla}\cdot\left(c\vec{\mathcal{F}}\right) = \vec{\nabla}\cdot\left(\frac{\partial\vec{E}}{\partial t}\right)$$

which tells us

$$\vec{\mathcal{F}} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} .$$

Ampere's law becomes

$$\vec{\nabla}\times\vec{B} = \frac{4\pi\vec{J}}{c} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t} .$$

Substituting this in with our other equations yields the *Maxwell equations*:

$$\begin{aligned} &\text{Gauss's law:} && \vec{\nabla}\cdot\vec{E} = 4\pi\rho \\ &\text{Magnetic law:} && \vec{\nabla}\cdot\vec{B} = 0 \\ &\text{Faraday's law:} && \vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \\ &\text{Generalized Ampere's law:} && \vec{\nabla}\times\vec{B} = \frac{4\pi\vec{J}}{c} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t} . \end{aligned}$$

It was James Clerk Maxwell who first fixed this inconsistency and argued that the $\partial\vec{E}/\partial t$ term must be present in Ampere's law. Interestingly, he did not argue this on the basis of continuity, but rather purely on the grounds of symmetry. We'll return to this point later.

You should also be aware of the form of these equations in SI units[1] — all those cs and 4πs are replaced with various combinations of μ_0s and ϵ_0s:

$$\begin{aligned}
\text{Gauss's law:} \qquad & \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \\
\text{Magnetic law:} \qquad & \vec{\nabla}\cdot\vec{B} = 0 \\
\text{Faraday's law:} \qquad & \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \\
\text{Generalized Ampere's law:} \qquad & \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}\ .
\end{aligned}$$

These four equations *completely* summarize 8.022. Every phenomenon in electricity and magnetism can be derived from these equations. Many of our most important tools for various analyses come from the *integral* version of these equations, which we will discuss shortly.

19.3 The displacement current

What does this new term mean? It turns out that it actually has a very nice, physical interpretation. We can begin to understand this using the old trick of dimensional analysis: electric field has the units of (charge)/(length)^2. The rate of change of electric field has units of (charge)/[(time)(length)^2]. This is the same thing as (current)/(length)^2, which is current density. The generalized Ampere's law is thus often written

$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\left(\vec{J} + \vec{J}_d\right)$$

where

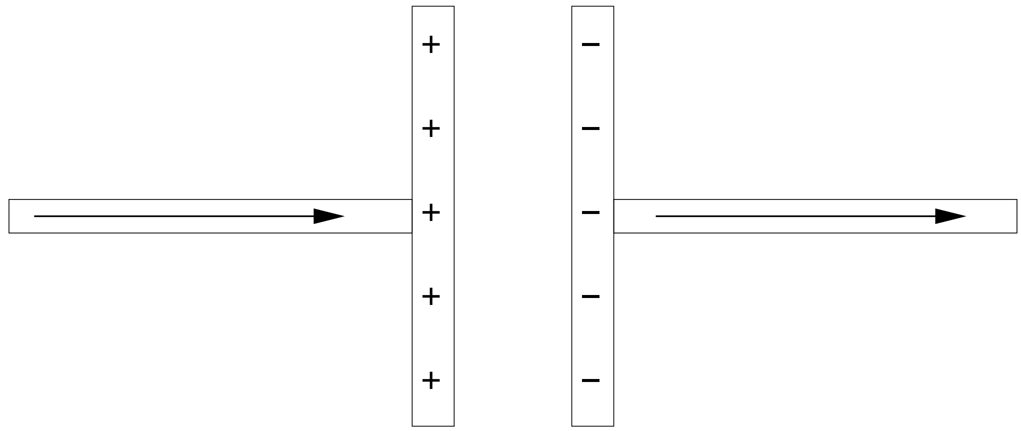
$$\vec{J}_d = \frac{1}{4\pi}\frac{\partial\vec{E}}{\partial t}$$

is known as the *displacement current*, or, more correctly, the displacement current *density*.

The displacement current is not a "real" current, in the sense that it does not describe charges flowing through some region. However, it *acts* just like a real current. Whenever we have a changing $\vec{E}$ field, we can treat its effects as due to the displacement current density arising from that field's variations.

[1] Note that there's a serious typo in Purcell's listing of these equations in SI units, Eq. (15') of Chapter 9.

In particular, the displacement current helps to fix up a few subtleties in circuits that might have bothered you. (I know for a fact that some students have wondered about these issues, and have asked some very good questions!) Consider a current flowing down a wire and charging up a capacitor:

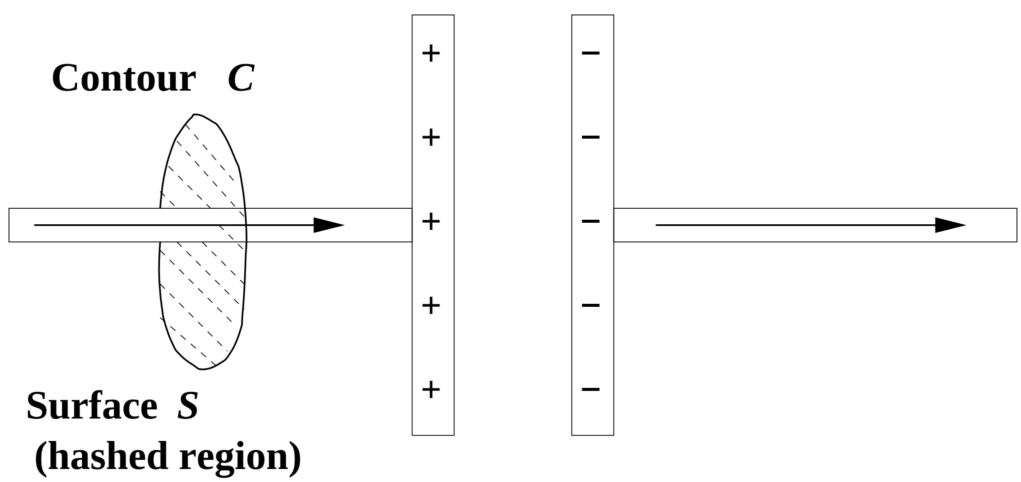


The old integral formulation of Ampere's law (which can be derived from the differential form) tells us

$$\begin{aligned} \oint_C \vec{B} \cdot d\vec{s} &= \frac{4\pi}{c} I_{\rm encl} \\ I_{\rm encl} &= \int_S \vec{J} \cdot d\vec{a} \, . \end{aligned}$$

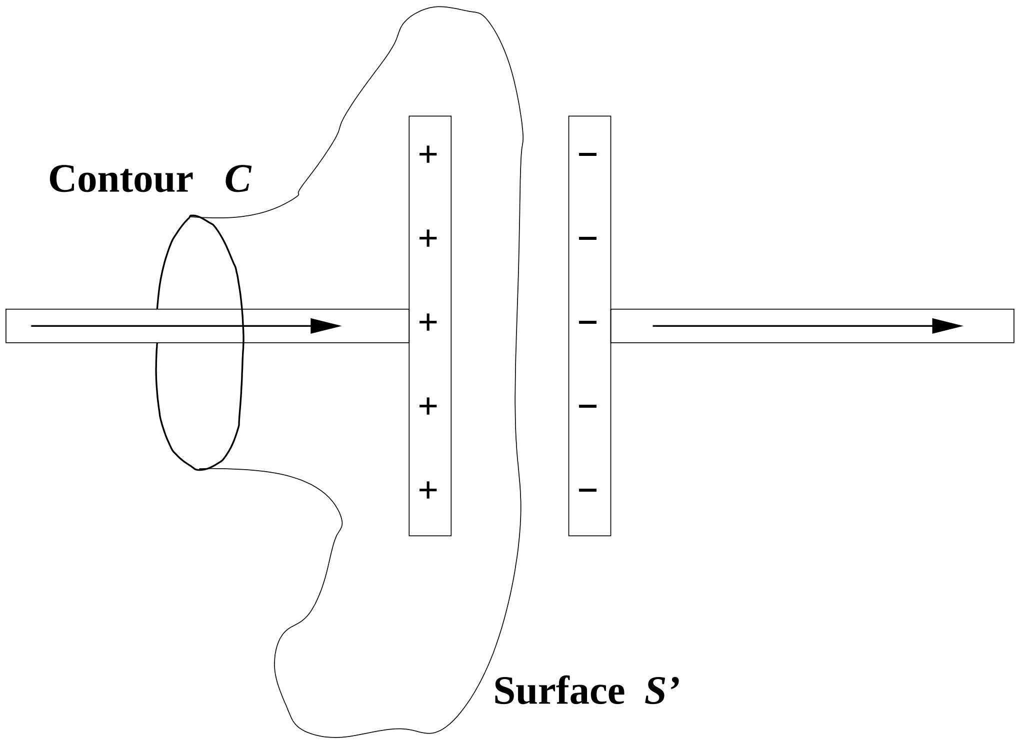
In words, the line integral of the magnetic field around a closed contour C equals $4\pi/c$ times the current enclosed by that contour. The current enclosed by that contour is given by integrating the flux of the current density through a surface S which is bounded by C.

We've normally used this law in the following way: we make a little "Amperian loop" around one of our wires. The contour C is this loop; the surface S is the disk for which S is the border:



The current $I_{\rm encl}$ is then just the current I flowing into the capacitor. Nice, simple, logical.

We now make this complicated. To go between the differential form of Ampere's law and this integral form of it, we use Stoke's theorem. A key fact of using this theorem is that the surface S can be ANY surface that has C as its boundary. In particular, we can choose this surface, S':



The amount of current flowing through this surface is ZERO: no charge flows between the plates! However, there is a flux of *displacement current* between the plates. When we turn our "generalized Ampere's law" into an integral form, we get

$$\begin{aligned}
\oint_C \vec{B} \cdot d\vec{s} &= \frac{4\pi}{c}\left(I_{\rm encl} + I_{d,\ \rm encl}\right) \\
\text{where} \qquad I_{d,\ \rm encl} &= \int_S \vec{J}_d \cdot d\vec{a} \\
&= \frac{1}{4\pi}\int_S \frac{\partial \vec{E}}{\partial t} \cdot d\vec{a} \\
&= \frac{1}{4\pi}\frac{\partial}{\partial t}\int_S \vec{E} \cdot d\vec{a} \\
&= \frac{1}{4\pi}\frac{\partial \Phi_E}{\partial t} \,.
\end{aligned}$$

The displacement current flowing through a surface S (as opposed to the displacement current density) is simply related to the rate of change of electric flux through that surface.

It is simple to show that, for the goofy surface S' drawn above, this rate of change of electric flux is exactly what we need to fix up Ampere's law. First, note that the electric field between the plates points in the same direction as the current; call that direction $\hat{x}$. At any given instant, the electric field between the plates (approximating it to be perfectly uniform) is given by

$$\vec{E} = \frac{4\pi Q}{A}\hat{x}$$

where A is the area of the plates. The electric flux between the plates is

$$\Phi_E = 4\pi Q$$

— not too surprising, given Gauss's law! The rate of change of this flux is

$$\frac{\partial \Phi_E}{\partial t} = 4\pi\frac{\partial Q}{\partial t} = 4\pi I\ .$$

(Since we are charging up this capacitor, $I = dQ/dt = \partial Q/\partial t$. Partial derivatives and total derivatives are the same since the charge only depends on time.) The displacement current "flowing" between the plates is thus

$$I_{d,\ \mathrm{encl}} = \frac{1}{4\pi}\frac{\partial \Phi_E}{\partial t} = I\ .$$

This tells us that

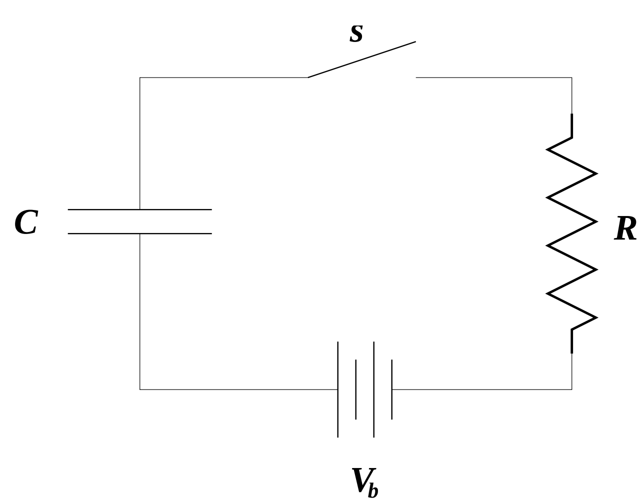
$$\int_S \vec{J}\cdot d\vec{a} = \int_{S'} \vec{J}_d \cdot d\vec{a} = I\ .$$

No matter which surface we use, we find

$$\oint_C \vec{B}\cdot d\vec{s} = \frac{4\pi I}{c}\ .$$

That's good!

Before moving on, note that the displacement current fixes up a somewhat bizarre aspect of circuits with capacitors that I know had some people a tad confused. When we analyze a circuit like this,



we claim that the same current $I(t)$ flows through all the circuit elements. However, we know that this *cannot* be totally true — there is no way for "real" current to make it across the plates of the capacitor. However, *displacement* current certainly makes it across! The displacement current plays the role of continuing[2] the "real" current through the gap in the capacitor. As such, it plays a very important role in ensuring that our use of Kirchhoff's laws is valid.

19.3.1 Integral form of Maxwell's equations

To wrap this section up, let's write down the four Maxwell equations in integral form. These are easily derived by plugging the differential forms into integrals and invoking various vector theorems; hopefully, such manipulations are close to second nature for you now.

$$\text{Gauss's law:} \qquad \Phi_E(S) = \oint_S \vec{E} \cdot d\vec{a} = 4\pi Q_{\rm encl}\ .$$

In words: **The electric flux through a closed surface S equals 4π times the charge enclosed by S.**

$$\text{Magnetic Gauss's law:} \qquad \Phi_B(S) = \oint_S \vec{B} \cdot d\vec{a} = 0\ .$$

In words: **The magnetic flux through a closed surface S equals 0**. We've never actually derived this, but it follows quite naturally from our earlier discussion of Gauss's law (no magnetic point charges!) and the rule $\vec{\nabla} \cdot \vec{B} = 0$.

$$\text{Faraday's law:} \qquad \mathcal{E} = \oint_C \vec{E} \cdot d\vec{s} = -\frac{1}{c}\frac{\partial \Phi_B}{\partial t} = -\frac{1}{c}\frac{\partial}{\partial t}\int_S \vec{B} \cdot d\vec{a}\ .$$

In words: **The EMF induced around a closed contour C equals $-1/c$ times the rate of change of the magnetic flux through the surface S bounded by C.**

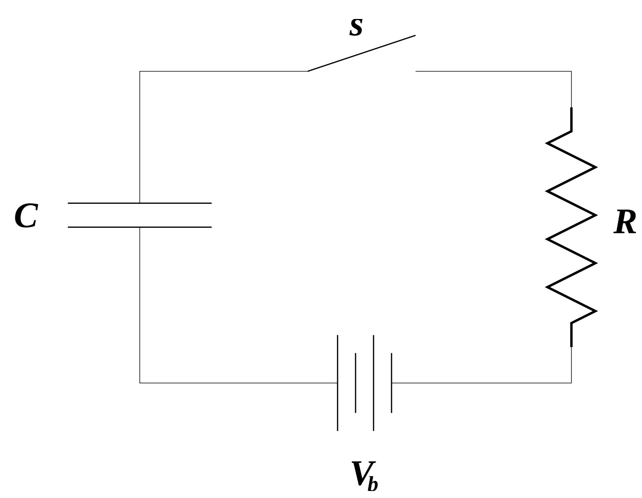
$$\begin{aligned} \text{Generalized Ampere's law:} \qquad & \oint_C \vec{B} \cdot d\vec{s} = \frac{4\pi}{c}\,(I + I_d) \\ \text{where} \qquad & I = \int_S \vec{J} \cdot d\vec{a} \\ & I_d = \int_S \vec{J}_d \cdot d\vec{a} = \frac{1}{4\pi}\frac{\partial \Phi_E(S)}{\partial t}\ . \end{aligned}$$

In words: **The line integral of the magnetic field around a closed contour C equals $4\pi/c$ times the sum of the total current — real plus displacement — passing through that contour**.

[2]Indeed, in many older textbooks, the phrase "continuation current" is found instead of displacement current, reflecting the fact it continues the real current.

19.4 Example: Using the displacement current

The beautiful thing about the displacement current is that we can now treat it just like a regular current. Consider this example: suppose we have an RC circuit, and we charge up the capacitor:



As the capacitor charges up, a displacement current flows between the plates. As a consequence, a *magnetic* field is induced between the plates, in addition to the electric field that is building up. We'll now calculate this magnetic field.

The key thing we need is the electric field between the plates as a function of time. To keep things simple, let's assume the plates are circular, with radius a, and we'll assume that they are close enough that we can ignore edge effects. Then, the electric field between the plates as a function of time is

$$E(t) = \frac{4\pi Q(t)}{\pi a^2} .$$

The displacement current *density* is

$$\begin{aligned} J_d(t) &= \frac{1}{4\pi}\frac{\partial E}{\partial t} = \frac{1}{\pi a^2}\frac{\partial Q}{\partial t} \\ &= \frac{I(t)}{\pi a^2} . \end{aligned}$$

Note that this is spatially constant, though it varies with time.

This circuit is something we beat to death long ago, so we can just quote the results for the current we found then:

$$I(t) = \frac{V_b}{R} e^{-t/RC} .$$

We're now ready to compute magnetic fields inside the plates. Since no "real" current flows there, our generalized Ampere's law reduces to

$$\oint_C \vec{B} \cdot d\vec{s} = \frac{4\pi}{c} \int_S \vec{J}_d \cdot d\vec{a} .$$

Let's pick as our contour C a circular path of radius $r < a$; the simplest surface S is then the disk of radius r. Then,

$$\oint_C \vec{B} \cdot d\vec{s} = 2\pi r B(r) \ .$$

For the right hand side, we need

$$\frac{4\pi}{c} \int_S \vec{J}_d \cdot d\vec{a} = \frac{4\pi I(t) r^2}{ca^2} \ .$$

The magnetic field is thus

$$B(r) = \frac{2rI(t)}{ca^2} = \frac{2rV_b}{cRa^2} e^{-t/RC} \ .$$

Direction is of course given by right-hand rule: point your right thumb parallel to $\vec{J}_d$ and your fingers curl in the same sense $\vec{B}$.

19.5 Source-free Maxwell's equations

As we shall see shortly, an extremely important limit of Maxwell's equations is found when there are no sources: $\rho = 0$, $\vec{J} = 0$. The equations become

$$\begin{aligned}
\vec{\nabla} \cdot \vec{E} &= 0 \\
\vec{\nabla} \cdot \vec{B} &= 0 \\
\vec{\nabla} \times \vec{E} &= -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \\
\vec{\nabla} \times \vec{B} &= \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \ .
\end{aligned}$$

These equations are almost perfectly symmetric! Indeed, when Maxwell first introduced the new term in the "generalized Ampere's law", this was his major motivation: he felt that there must be an underlying symmetry between electric and magnetic fields. It is a mark of his remarkable insight that this feeling was entirely correct.

Even more significantly, these equations point to an extremely interesting and important coupling between the magnetic and electric fields. If we imagine we have a time varying electric field, this shows that it will "source" some kind of magnetic field. That magnetic field will be time varying, and so it will source an electric field. That electric field will then source a magnetic field, which will source an electric field, which will ...

You get the point. We end up with an infinite chain of electric to magnetic to electric to ... This is *radiation*. It will be our next major topic.