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LECTURE 24:
A (VERY) BRIEF INTRODUCTION TO GENERAL RELATIVITY.

24.1 Gravity?

The Coulomb interaction between two point charges looks essentially identical to the gravitational interaction between two masses. Does this mean that everything we have done so far can also be used to describe gravitational interactions (perhaps with some slight modifications)?

The answer to this turns out to be *almost* — but not quite. To understand what this means in detail, we need to do a little bit of background work. The key point to be made here is that Maxwell's equations and electricity and magnetism are relativistically correct: all of E&M "knows about" special relativity, in the sense that the equations make sense in any frame of reference. (When we transform reference frames, we modify the $\vec{E}$ and $\vec{B}$ fields according to those relativistic transformation laws we all learned to love ages ago. However, the new fields *still* satisfy Maxwell equations.) This is *NOT* the case for gravity! Significant modifications need to be made to make gravity relativistic. What we end up with is Einstein's theory of *general relativity.* In this lecture we will look at some basic properties of general relativity.

To begin, we need to introduce a bit of notation.

24.2 4-vectors

Since space and time are unified in relativity, it doesn't make much sense to treat them separately. Accordingly, many concepts are described using *4-vectors*, quantities that are essentially just like the vectors we've known and loved since kindergarten, but with an extra "timelike" component. For example, the position 4-vector is written

$$x^\mu = (ct, x, y, z) \ .$$

The way to read this is that μ is an *index*, ranging from 0 to 3: $x^0 = x^{\mu=0} = ct$; $x^1 = x^{\mu=1} = x$; $x^2 = x^{\mu=2} = y$; $x^3 = x^{\mu=3} = z$. Whenever you see an x with a number superscript, it means the index — it does NOT mean take it to a higher power! This system can be a little confusing when you first encounter it.

That the "timelike component of position" is time makes perfect sense. What do we get when we try to make something like momentum into a 4-vector? It turns out that the "timelike component of momentum" is the *energy*:

$$p^\mu = (E/c, p^x, p^y, p^z) \ .$$

We'll introduce a few other 4-vectors before we're done here.

24.2.1 Invariant interval

Back on Pset 6, we showed that the quantity

$$\Delta s^2 = -c^2\Delta t + \Delta x^2 + \Delta y^2 + \Delta z^2$$

is *invariant*: **all** inertial observers agree on its value. This can be written this in terms of a quantity called the *metric*:

$$\Delta s^2 = \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} g_{\mu\nu}\Delta x^\mu \Delta x^\nu$$

where the *metric tensor* $g_{\mu\nu}$ can be represented as a 4×4 matrix:

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

This relation is usually written using the *Einstein summation convention*: repeated indices are *assumed* to be summed from 0 to 3. In this convention, the invariant interval then becomes

$$\Delta s^2 = g_{\mu\nu}\Delta x^\mu \Delta x^\nu \ .$$

We will use this notation from now on.

24.2.2 Covariant versus contravariant

The 4-vectors that we have introduced so far are, technically speaking, written in terms of *contravariant* components. Using the metric, we define a new version of our 4-vector:

$$\begin{aligned} x_\mu &\equiv g_{\mu\nu}x^\nu \\ &= (-ct, x, y, z) \ . \end{aligned}$$

The effect is just to change the sign of the timelike component.

It should be noted that there is *no* physics in this operation. The difference between "index up" (contravariant) and "index down" (covariant) is just needed to make sure that the math works out right. In particular, in writing down quantities that matter for physics, you typically end up summing over repeated indices. When we do this, the index has to be down on one quantity and up on the other.

24.2.3 Derivatives

Finally, one handy notational definition: we write

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} \ , \qquad \partial^\mu \equiv \frac{\partial}{\partial x_\mu} \ .$$

There's absolutely nothing deep about this! However, it makes certain quantities look nice.

Notice that $\partial_0 = (1/c)\partial/\partial t = -\partial^0$.

24.3 Electric & magnetic fields?

Can we make a 4-vector that incorporates electric and magnetic fields? With a little thought, we see that the answer to this has to be **no** — the three components of electric fields and the three components of magnetic fields are 6 separate numbers. A 4-vector — pretty much by definition! — only has 4 separate numbers.

What we *can* do is make a 4-vector out of the magnetic vector potential. We use the electrostatic scalar potential as the time component, and find

$$A^\mu = (\phi, A^x, A^y, A^z) \; .$$

(Note that in cgs, ϕ and $\vec{A}$ have the same units.) How do we get fields from this potential? We take a derivative! The one that we want turns out to be

$$F^{\mu\nu} \equiv \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu} \; .$$

This tensor contains EXACTLY the quantities that we want! When we write out the tensor $F_{\mu\nu}$, we see that its components are given by

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \; .$$

The tensor is often called the *Faraday tensor*.

Although it's not so simple to see where this one comes from, there's a second tensor we can construct in a similar manner, often just called the *Dual Faraday* tensor:

$$^*F^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z & E_y \\ -B_y & E_z & 0 & -E_x \\ -B_z & -E_y & E_x & 0 \end{pmatrix}$$

We'll discuss the point of introducing this "dual" form in just a moment.

A second 4-vector we can naturally make is for the source: we combine the vector current density with charge density ρ to make

$$J^\mu = (\rho c, J^x, J^y, J^z) \; .$$

Using this, we can write Maxwell's equations in a very simple form:

$$\partial_\nu F^{\mu\nu} = \frac{4\pi}{c} J^\mu \; , \qquad \partial^*_\mu F^{\mu\nu} = 0 \; .$$

Expand one example: The $F^{\mu\nu}$ equation for $\mu = 0$. Then,

$$\begin{aligned} & \frac{\partial F^{0\nu}}{\partial x^\nu} = \frac{4\pi}{c} J^0 \\ & \frac{\partial F^{0x}}{\partial x} + \frac{\partial F^{0y}}{\partial y} + \frac{\partial F^{0z}}{\partial z} = \frac{4\pi}{c}(c\rho) \\ & \frac{\partial E^x}{\partial x} + \frac{\partial E^y}{\partial y} + \frac{\partial E^z}{\partial z} = 4\pi\rho \\ \longrightarrow \qquad & \vec{\nabla} \cdot \vec{E} = 4\pi\rho \end{aligned}$$

If you expand and combine the $F^{\mu\nu}$ equation for $\mu = 1, 2, 3$, you get

$$
\begin{aligned}
& -\frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{J} \\
\longrightarrow \qquad & \vec{\nabla} \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c}\vec{J} \, .
\end{aligned}
$$

Likewise, the $\mu = 0$ part of the ${}^*F^{\mu\nu}$ equation gives us

$$\vec{\nabla} \cdot \vec{B} = 0 \; ;$$

the $\mu = 1, 2, 3$ parts give us

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \; .$$

24.4 General relativity

We've now got enough background that we can now sketch the general theory of relativity. Without going into too much detail the key quantity we need to describe gravity in a relativistic manner is the metric: our starting point is the equation

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \; .$$

Rather than being constant, as it was in special relativity, the metric is a varying function in general relativity. This turns out to mean that spacetime has *curvature*. All of gravity is encoded in the mathematics of this curvature.

The metric plays the same role for gravity that the 4-vector A^μ plays in electricity and magnetism — think of it as a "potential" for gravity. The equivalent of Maxwell's equations are the *Einstein Field Equations*:

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \; .$$

Some explanation of this equation is in order. On the right-hand side, the tensor $T_{\mu\nu}$ — called the "stress-energy tensor" — describes the flow of 4-momentum. This tensor plays a role in these equations analogous to that of J^μ in Maxwell's equations. A component $T_{\mu\nu}$ describes the flow of the μ component of 4-momentum in the ν direction, in the same way that J^μ describes the flow of charge in the μ direction. For example, T_{00} describes the flow of the zero component of 4-momentum into the time direction; this turns out to be the energy density. Components like T_{0x} describe the flow of energy in the x direction. This is related to the mass density ρ_m and the velocity with which mass flows.

The tensor $G_{\mu\nu}$ is, in general, an enormous mess: it is found by applying a 2nd order, coupled nonlinear differential operator to the metric $g_{\mu\nu}$. The key thing to note is that, roughly speaking, it corresponds to *two* derivatives of the metric. Thus Einstein's equations are of the form "two derivatives of the potential (metric) equals the source", exactly as Maxwell's equations are.

24.4.1 Gravitational "forces"

Before moving on, we should talk about gravitational forces. In electricity and magnetism, the force that operates upon charges is

$$\vec{F}_{\text{E\&M}} = q\left[\vec{E} + (\vec{v}/c) \times \vec{B}\right] .$$

In general relativity, it turns out that there is *no such thing as a gravitational force!* Instead, what happens is that bodies move along a trajectory of *extremal time*. To understand what this means, go back to the definition of the invariant interval:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu .$$

For a body following a trajectory through spacetime, $ds^2 = -c^2 d\tau^2$, where τ describes the time as measured on a clock on that body. The path that the body follows through spacetime and the gravitational "forces" that act on it are found by working out the extremum of the total time τ that accumulates during its travels. The details of that calculation are somewhat beyond 8.022 — suffice it to say that it is a well-founded calculation.

An analogy helps to explain what is going on. Suppose we're on a flight from Boston to Tokyo. To save fuel, we want the trip to be as short as possible. The shortest path between two points is a straight line. But, on a round globe, what does "straight line" really mean? We go back to the old fashioned definition we learned in kindergarten: a "straight line" is the path which gives the shortest distance between two points. This is why a flight from Boston to Tokyo might actually go over the North Pole.

On a map, this flight might look crazy — it looks like you're bending the trajectory of the plane in as stupid a way possible. A person who believes in the reality of maps — as opposed to the reality of the earth — might say that the reason for this crazy trajectory is that "forces" act on the plane in such a way that the path over the North Pole is "easier" to follow than a simple logical "straight" line (straight according to the map, that is).

In general relativity, gravitational "forces" are like the fictional forces to which our map worshipper ascribes the airplane's trajectory.

24.5 A special metric

It turns out that the spacetime metric outside of a rotating body is given by

$$ds^2 = -(1 + 2\Phi)(c^2 dt^2) + (dx - \beta^x dt)^2 + (dy - \beta^y dt)^2 + (dz - \beta^z dt)^2 .$$

(Strictly speaking, this is an approximate result, only holding under certain conditions — namely that the body can't be too dense, and that its internal velocities must be much less than the speed of light.) The quantities β^x, β^y, and β^z are components of a vector $\vec{\beta}$ that has units of velocity.

If we run this metric through our procedure to find maximal time, we find that the "force" that acts on a body is given by

$$m\frac{d^2\vec{x}}{dt^2} = m\vec{g} + m(\vec{v}/c) \times \vec{H} ,$$

where

$$\vec{g} = -\vec{\nabla}\Phi , \qquad \vec{H} = \vec{\nabla} \times \vec{\beta} .$$

Notice that this force law looks *EXACTLY* like an electromagnetic force! In general relativity, the gravitational force looks just like the normal Newtonian force — encapsulated in the gravitational field $\vec{g}$ — is supplemented by a *gravitomagnetic* force arising from a field $\vec{H}$ that acts just like a magnetic field.

If we run this metric through Einstein's field equations, we find

$$\begin{aligned} \vec{\nabla}\cdot\vec{g} &= -4\pi G\rho_m \qquad \text{(Gravitational Gauss's law)} \\ \vec{\nabla}\cdot\vec{H} &= 0 \\ \vec{\nabla}\times\vec{g} &= 0 \\ \vec{\nabla}\times\vec{H} &= -\frac{16\pi G\vec{J}_m}{c} \qquad \text{(Gravitational Ampere's law)} \\ \text{where} \qquad \vec{J}_m &= \rho_m\vec{v}\,. \end{aligned}$$

The equations which fix the gravitational fields $\vec{g}$ and $\vec{H}$ look almost EXACTLY like Maxwell's equations! There are some important differences. Notice that the signs are reversed compared to the Maxwell equations. This is because gravity is *always* an attractive force. If the force between two masses were repulsive, we would find a + sign for the $\vec{\nabla}\cdot\vec{g}$. There is also a factor of 4 difference on the Ampere's law. This is harder to explain; it is related to the fact that the gravitational "potential" is a 2 index tensor, rather than a 1 index vector.

This analogy continues. If we move a gyroscope with an angular momentum vector $\vec{s}$ through this spacetime, it *precesses*: the vector feels a torque according to

$$\frac{d\vec{s}}{dt} = \frac{1}{2}\vec{s}\times\vec{H}$$

— an equation that is very similar to the torque felt by a magnetic dipole in a magnetic field, $\vec{\tau} = \vec{m}\times\vec{B}$.

These relativistic effects are ordinarily quite small and don't play much of an important role. Nonetheless, they are **not** zero. Last year, NASA launched a satellite to try to measure this "gravitomagnetic precession". This satellite, called *Gravity Probe B*[1] consists of an extremely precise and sensitive gyroscope that *very* slowly changes the direction of its angular momentum due to the gravitomagnetic field induced by earth's rotation. The net effect accumulated effect is about 41 milliarcseconds (1.1×10^{-5} degrees) *per year*. It's quite an amazing technological feat to be able to measure such a puny effect!

These effects play a much more robust role when gravitational fields and rotation speeds are stronger. In this case, the metric we used above is not really all that accurate. At least qualitatively, however, this picture of a Newtonian-like gravitational field $\vec{g}$ and a gravitomagnetic field $\vec{H}$, remains useful. The extra "force" arising from the gravitomagnetic fields are extremely strong and important! We look for such effects when we study objects in orbit around neutron stars and black holes — we *use* this gravitomagnetic effect as a means of learning about things in the universe.

[1] Gravity Probe A tested the fact that clocks run slow in gravitational fields.