Massachusetts Institute of Technology
Department of Physics
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Lecture 10
Describing matter in bulk

10.1 A box full of dust

In our discussion so far, we have discussed how to analyze the kinematics of *particles* — pointlike entities with velocity and mass, momentum and energy. The focus on particles is an important step to making our laws of physics comport with the principle of Lorentz covariance. However, a lot of the matter that we study in physics isn't in a form that we study particle by particle, but instead is distributed in bulk over some volume. Various aspects of the properties of this bulk matter vary according to the reference frame in which it is observed. In today's lecture, we will introduce tools that are used to characterize bulk matter, and will examine what properties of the characterization change as we change frames.

Begin by considering a box full of *dust*. "Dust" is how we describe matter that doesn't interact with itself — it doesn't exert pressure or do anything interesting other than take up space. Think of it as a pile of particles with mass, but no other interesting property[1].

We begin with the simplest way to characterize this matter: we take the box to be at rest with respect to us, and we count the number of dust particles it contains. We find that the box contains N particles, and that the box has a volume of V. Then, we say that the dust has a *number density*

$$n_0 = N/V \ . \tag{10.1}$$

The number n_0 characterizes perhaps the most important characteristic of the dust, given what we know about it so far. (The reason for the "0" subscript will be made clear in a moment.) Note the dimensions of n_0: number per unit volume, or number per length cubed.

Now take the box full of dust to be, as we observe it, in motion. What is different from the rest frame view? What is the same?

The total number of dust particles must be the same — simply making the box move cannot create or destroy any of the dust. So the number of particles N is independent of the frame in which we measure it. But, one of the linear dimensions of the box is contracted by a factor γ. This reduces the volume of the box by a factor γ according to our measurements, which in turn means that the number density must increase by a factor γ:

$$n = \gamma N/V = \gamma n_0 \ . \tag{10.2}$$

We will use n to stand for the number density that we measure in our frame of reference. This reduces to n_0 if our frame of reference happens to be the dust's own rest frame.

When we observe the dust to move, it acquires one other property: some volumes which were empty of dust at time t will contain dust a time $t + \Delta t$ later; other volumes that

[1]Such "dust" doesn't really exist — any dust that we encounter in reality is more interesting than the dust we use in this lecture. Our dust is an idealization that we use to formulate the framework that we are working in, and serves as a useful starting point. Once we've developed a framework for this more-or-less fictional idealization, we can add more features and properties, pushing it toward something realistic.

contained dust will lose it. This is because the dust is now flowing: there is a *flux* of dust. Suppose that we measure the volume[2] to have length L. Let's orient our coordinate axes so that the box is moving in the x direction, and we can write $\mathbf{v} = v\mathbf{e}_x$. At time $t = 0$, the back of the box is at $x = 0$, and the front of the box is at $x = L$. The cross section of the box has area A (so that $V = AL$).

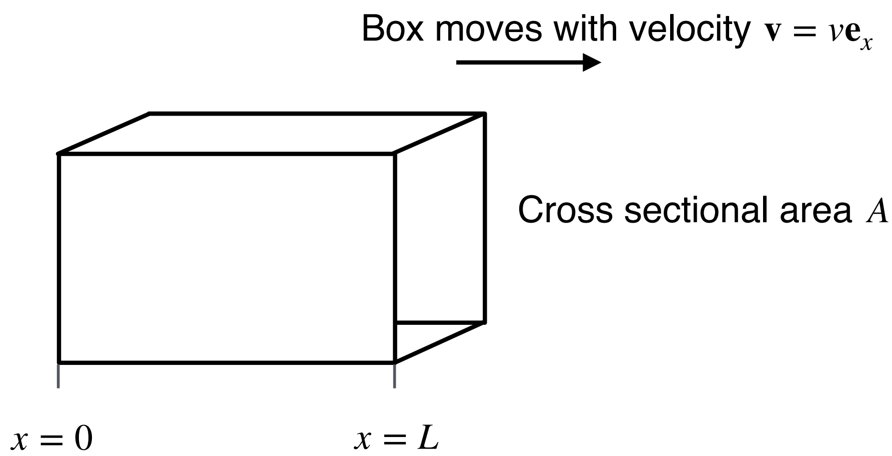

Figure 1: Box as described in the text, at time $t = 0$.

At time $t = \Delta t$, the volume from $x = 0$ to $x = v\Delta t$ has been emptied of dust; the volume from $x = L$ to $x = L + v\Delta t$ has filled up with dust. The box is gaining $nAv\Delta t$ dust particles at the front end, and losing $nAv\Delta t$ dust particles at the back end. Dividing by $A\Delta t$, we the rate at which dust is entering one end per unit cross section area is

$$\frac{dN}{dA\,dt} = nv \; . \tag{10.3}$$

The same rate is leaving the box at the back end.

Equation (10.3) defines a *flux* of particle number into and out of the box. Let's make this a bit more general: we define the x component of the *number flux 3-vector* by

$$n^x = nv = \gamma n_0 v \; . \tag{10.4}$$

You should be able to convince yourself that there was no reason to restrict ourselves to dust moving in the x direction, and we can define a general number flux 3-vector as

$$\mathbf{n} = n\mathbf{v} = \gamma n_0 \mathbf{v} \; . \tag{10.5}$$

The number flux 3-vector $\mathbf{n}$ tells us the number of dust particles per unit area that crosses into (or out of) a region per unit time.

[2]Bear in mind that this means L is *not* the rest frame length of the box

Let's think about the flow of dust into or out of a region a little more carefully. Imagine that dust is flowing through our frame of reference, and that at each point in space it has a number density n and a number flux 3-vector $\mathbf{n}$. Imagine that both of these quantities can vary as a function of position and time: the flow of dust may bend and twist as it flows, with the amount in the flow rising and falling with time.

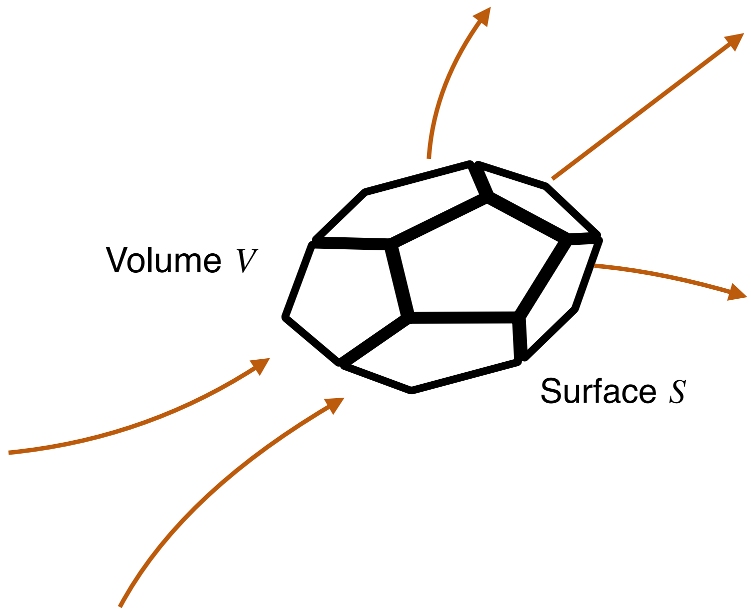

Figure 2: Dust with number flux 3-vector $\mathbf{n}$ flows into and out of a volume V which is bounded by a surface S.

Imagine that this "river" of dust flows into a volume V which is bounded by a surface S. In a time Δt, the change in the number of dust particles in the volume is given by

$$\begin{aligned} \Delta N &= -\Delta t \oint_S \mathbf{n} \cdot d\mathbf{A} \\ &= \int_V [n(t + \Delta t) - n(t)]\, dV \;. \end{aligned} \tag{10.6}$$

Let's deconstruct Eq. (10.6). On the first line, we have introduced and are using the outward directed area element $d\mathbf{A}$. This is a differential of area to which we assign a direction: It points in the out direction, normal (orthogonal) to the surface. The minus sign is because the area element is outward pointing: if $\mathbf{n} \cdot d\mathbf{A} < 0$, then dust is flowing into the volume and ΔN is positive; vice versa if $\mathbf{n} \cdot d\mathbf{A} > 0$.

To write down the second line, note that ΔN is the change in total number contained by the volume. We get this total number by integrating the number density over the volume V; its change is given by subtracting the amount that was there at time t from the amount that is there a time Δt later.

Next, divide both sides by Δt. We can take Δt inside the integral, yielding

$$\int_V \frac{[n(t + \Delta t) - n(t)]}{\Delta t}\, dV = -\oint_S \mathbf{n} \cdot d\mathbf{A} \;. \tag{10.7}$$

Taking the limit $\Delta t \to 0$,

$$\int_V \frac{\partial n}{\partial t}\, dV = -\oint_S \mathbf{n} \cdot d\mathbf{A} \,. \tag{10.8}$$

For the next step, we invoke the *divergence theorem*: for any 3-vector $\mathbf{F}$ defined over a region V that has a closed surface S,

$$\oint_S \mathbf{F} \cdot d\mathbf{A} = \int_V (\nabla \cdot \mathbf{F})\, dV \,. \tag{10.9}$$

Applying the divergence theorem on the right-hand side of Eq. (10.8) and then moving it to the left, we have

$$\int_V \left[\frac{\partial n}{\partial t} + \nabla \cdot \mathbf{n}\right] dV = 0 \,. \tag{10.10}$$

This equation must hold no matter what V we use. The only way for that to be the case is if the term in square brackets in Eq. (10.10) vanishes. This means that the number density n and the number flux $\mathbf{n}$ are related by the *continuity equation*

$$\frac{\partial n}{\partial t} + \nabla \cdot \mathbf{n} = 0 \,, \tag{10.11}$$

or, expanding out the components in the divergence term,

$$\frac{\partial n}{\partial t} + \frac{\partial n^x}{\partial x} + \frac{\partial n^y}{\partial y} + \frac{\partial n^z}{\partial z} = 0 \,. \tag{10.12}$$

Everything we have done can be organized into a particularly tidy package using 4-vectors. First, note that Eqs. (10.2) and (10.5) have *almost* exactly the form of the components of a 4-vector: we treat (10.2) as the timelike component, and then (10.5) defines the spatial components. The only reason this doesn't quite work is that (10.2) has the wrong dimensions: it is number per unit volume, whereas the components in (10.5) have the dimensions number per unit area per unit time.

This is easily fixed: just multiply (10.2) by the speed of light c. Doing so, we define the *number flux* 4-vector $\vec{N}$, whose components are

$$\begin{aligned}(N^0, N^1, N^2, N^3) &= (nc, nv^1, nv^2, nv^3) \\ &= (\gamma n_0 c, \gamma n_0 v^1, \gamma n_0 v^2, \gamma n_0 v^3) \,. \end{aligned} \tag{10.13}$$

Notice that this is nothing more than

$$\vec{N} = n_0 \vec{u} \,, \tag{10.14}$$

where $\vec{u}$ is the 4-velocity with which we observe the dust to be moving. Let's look at the invariant we can build out of $\vec{N}$:

$$\vec{N} \cdot \vec{N} = n_0^2 \vec{u} \cdot \vec{u} = -n_0^2 c^2 \,. \tag{10.15}$$

This tells us that the number flux 4-vector is timelike. Taking the scalar product of $\vec{N}$ with itself yields the number density of the dust in its own rest frame, times $-c^2$.

The 4-vector $\vec{N}$ also allows us to write the continuity equation in a particularly tidy way. Recalling that $x^0 = ct$, we see that Eq. (10.12) can be written

$$\frac{\partial N^\alpha}{\partial x^\alpha} = 0 \,, \tag{10.16}$$

or

$$\partial_\alpha N^\alpha = 0 \,. \tag{10.17}$$

Notice that there are no free indices left over: we sum over α, with one in the upstairs position and one downstairs, yielding a Lorentz invariant quantity (in the case, the number 0 — certainly a quantity that all Lorentz observers agree on). By setting everything up using 4-vectors, we have a *covariant* formulation of the continuity equation. If we have measured the 4-components of $\vec{N}$ in the frame of $\mathcal{O}$, and would like to know how they will appear in the frame of $\mathcal{O}'$, we simply apply a Lorentz transformation: $N^{\mu'} = \Lambda^{\mu'}{}_\alpha N^\alpha$. This quantity will obey the continuity equation provided we take derivatives using the coordinates $x^{\mu'}$ which are used by $\mathcal{O}'$: they will find $\partial_{\mu'} N^{\mu'} = 0$.

10.2 A box full of charge

This discussion of number continuity may have reminded you of a calculation that you did in electricity and magnetism. Suppose each grain of dust carries an electric charge q. Then, our calculation proceeds essentially exactly as before, but we can now look at the *charge density* associated with a volume, and we can think about a charge flux 3-vector, better known as the *current density*. Let's quickly see what this looks like.

If the number density of the dust in some frame is n, and if each dust grain carries a charge q, then the charge density ρ_q is given by

$$\rho_q = nq \,. \tag{10.18}$$

(We will use ρ for something different in a moment, hence the q subscript to denote charge density.) If these dust grains have a number flux 3-vector $\mathbf{n}$, then the flow of the dust carries a current density

$$\mathbf{J} = q\mathbf{n} = \rho_q \mathbf{v} \,. \tag{10.19}$$

Going through the derivation of number continuity again, but now including a charge q on each dust grain, yields the continuity equation for electric charge:

$$\frac{\partial \rho_q}{\partial t} + \nabla \cdot \mathbf{J} = 0 \,. \tag{10.20}$$

We can build a 4-vector out of this by defining its "zeroth" component using the charge density and the speed of light. We thus define $\vec{J}$ with components

$$(J^0, J^1, J^2, J^3) = (\rho_q c, J^1, J^2, J^3) \,. \tag{10.21}$$

With this formulation, we can write the equation of charge continuity as

$$\partial_\alpha J^\alpha = 0 \,. \tag{10.22}$$

We will return to this 4-vector shortly when we examine how to write the equations of electrodynamics in a way that makes their Lorentz covariance clear.

10.3 A box full of dust, revisited

Finally, let's give each dust grain a rest mass m. We could define a rest mass density $\rho_m = Nm/V$. However, we know that as we change frames, the most interesting quantities which describe a massive object are its energy γmc^2 and its momentum $\mathbf{p} = \gamma m\mathbf{v}$. So let's instead define the rest frame's *energy density* $\rho_0 = Nmc^2/V$. How does this quantity transform when we change frames?

Again, we know that we cannot create or destroy any dust grains, so N is the same in all frames. We also know that the length of the box along the relative motion of the frames is contracted by γ, so $V \to V/\gamma$. However, in this case, we also know that the energy of each dust grain is boosted by γ: the grain only had rest energy in the original rest frame, but it has both rest energy and kinetic energy in a frame moving with $\mathbf{v}$ relative to the rest frame. The energy density in this frame is given by

$$\rho = N(\gamma mc^2)/(V/\gamma) = \gamma^2 \rho_0 \,. \tag{10.23}$$

The fact that *two* powers of γ enter into this transformation law is interesting and important. When we carefully studied number density and charge density, we realized that these quantities were actually components of a 4-vector. If they had been Lorentz scalars, then they would have been invariants; the transformation would have involved no factors of γ. The number of dust grains in a box, or the total charge in a box, both fall into this category. When there is one factor of γ, that tells us that that we have stumbled onto a transformation law that involves one factor of the Lorentz transformation matrix $\Lambda^{\mu'}{}_{\alpha}$, and so the quantity we are looking at is a component of a rank-1 tensor — i.e., a 4-vector.

This factor of γ^2 tells us that the quantity we are examining is associated with a transformation law that involves *two* factors of the Lorentz transformation matrix. The quantity we are studying must a component of a rank-2 tensor — a quantity with two associated indices. Let us define

$$T^{\alpha\beta} = \frac{Nm}{V} u^\alpha u^\beta \qquad \text{or} \tag{10.24}$$

$$= p^\alpha N^\beta \,. \tag{10.25}$$

This quantity is known as the *stress-energy tensor*. The 00 or tt component describes energy density in some reference frame. To understand the other components, note the interpretation that Eq. (10.25) suggests: $T^{\alpha\beta}$ describes the flux of 4-momentum p^α in the direction of x^β. (Via Eq. (10.24), we see that $T^{\alpha\beta} = T^{\beta\alpha}$, so we can equally well call this the flux of 4-momentum p^β in the direction of x^α.) What do the other components mean? Let's go through these tensor looking at a couple of important groupings of components for this dust stress energy:

- $T^{00} = \gamma^2 n_0 mc^2$: As already discussed, this is *energy density*. Think of it as the density of p^0 flowing in the direction of x^0 — the flux of energy density through time.

- $T^{0i} = \gamma^2 n_0 mcv^i$: This is *energy flux*: the flow of the density of p^0 in the x^i direction. If you look carefully at the units, you'll see that this quantity is off a by factor with the units of velocity. More correctly, the energy flux is $T^{0i}c$. The root issue here is that 4-momentum component p^0 is E/c, so we need to correct with a factor of c. Correction factors like this don't change the essential physics.

- $T^{i0} = \gamma^2 n_0 mcv^i$: This is *momentum density.* Think of this as the density of momentum p^i flowing through time. Again, examining units, you'll see it's a bit off. More correctly, the momentum density is T^{i0}/c; the root issue here is that x^0 is c times t. (Needing to account for factors like this is one reason why many people use units in which $c = 1$. Keeping track of factors of c can become tiresome.) Notice that $T^{i0} = T^{0i}$ — using the relativistic definitions of energy and momentum, energy flux and momentum density are the same thing, modulo factors of c.

- $T^{ij} = \gamma^2 n_0 mv^i v^j$: This is *momentum flux*: the flow of momentum p^i in the x^j direction.

10.4 The stress-energy tensor more generally

Dust is a useful tool for introducing the stress-energy tensor and wrapping our heads around what the components of this tensor mean to a particular observer. However, dust is a somewhat limited class of matter. The stress-energy tensor is much broader than this. We conclude today's lecture by discussing the meaning of the stress-energy tensor as it is used to describe matter in general and, as we'll briefly discuss later, fields.

One often characterizes the stress-energy tensor by going into a frame of reference in which there is no bulk flow of material. For examine, if it is a fluid, this is the frame in which the fluid is a rest; such a frame is called "comoving" in this case. Note that a distribution of material might flow at different speeds in different places or at different times; think of this as how we characterize one small "element" of the material. In this frame, the different components take on exactly the meaning that we discussed for the components of the dust stress-energy tensor:

- T^{00} represents the energy density of the material.

- T^{0i} represents (modulo a factor of c) the energy flux of the material. Note that if no matter is actually moving, there still might be a flow of energy — the material might be conducting heat, or there may be radiation flowing in some direction.

- T^{i0} represents (modulo a factor of c) the momentum density of the material. Again, even if no matter is actually moving there can still be a density of momentum. Indeed, there *must be* momentum density if there is any flux of energy.

- T^{ij} represents the momentum flux. This 3×3 spatial tensor is important in its own right, and is known as the *stress tensor.* The on-diagonal and off-diagonal elements of the stress tensor deserve comment:

  - The on-diagonal elements (T^{xx}, T^{yy}, T^{zz}) tell us about the flow of momentum component p^i in the x^i direction. These components of the stress tensor tell us about the force (per unit area) the material exerts in the direction of its flow. When the material is a fluid, these components of the tensor describe *pressure.*
  - The off-diagonal elements (T^{xy}, T^{xz}, T^{yz}, plus symmetries) tell us about "non-normal" flows of momentum. In fluids, these terms are related to a property called its *viscosity*; it leads to forces along (i.e., parallel to) an interface, rather than normal to the interface (the way pressure operates).

An example of a material which is used in many analyses is a *perfect fluid*. It is a fluid for which there exists a frame of reference in which its stress-energy tensor has components $T^{\alpha\beta} = \mathrm{diag}(\rho, P, P, P)$, where ρ is the fluid's energy density, and P is its pressure.

The "perfect" in "perfect fluid" means that it represents a kind of Platonic ideal: there is no energy or momentum flow in a perfect fluid's rest frame (meaning that there is no heat conduction, or other mechanism to transport energy), and it has no viscosity. No viscosity means that if you were to dip your hand into it, none of the fluid would stick to you when you pulled your hand out. As such, the physics of perfect fluids has been mocked as the physics of "dry water."

The meaning of stress energy as a flux of 4-momentum allows us to derive a continuity equation for it. Let's reconsider Fig. 2, but rather than thinking about the flow of dust, think about the flow of 4-momentum. We then largely repeat our derivation of number continuity, but replace quantities related to number density with quantities related to 4-momentum density. In particular, let's replace the number density n with the 4-momentum density $T^{\alpha 0}$, and replace the number flux $n^i = nv^i$ with the 4-momentum flux $T^{\alpha i}$.

The total amount of 4-momentum in V is given by integrating $T^{\alpha 0}$ over this volume:

$$[p^\alpha(t)]_V = \frac{1}{c}\int_V T^{\alpha 0}(t)\, dV \ . \tag{10.26}$$

The factor of $1/c$ accounts for the fact that T^{00} is energy density, but p^0 is E/c, plus for the fact that T^{i0} has such a factor built into its definition. The *change* in the 4-momentum in V over an interval of time Δt is thus given by

$$\begin{aligned}\Delta p^\alpha &= \frac{1}{c}\int_V \left[T^{\alpha 0}(t+\Delta t) - T^{\alpha 0}(t)\right] dV \\ &= \Delta t \int_V \frac{\partial T^{\alpha 0}}{\partial x^0} dV \ .\end{aligned} \tag{10.27}$$

We can also account for the change by computing the flux of 4-momentum through the surface S bounding this volume during the time interval Δt:

$$\Delta p^\alpha = -\Delta t \oint_S T^{\alpha i} dA^i \ . \tag{10.28}$$

The minus sign is again because the area element $d\mathbf{A}$ (which has components dA^i) points outward from the volume. The divergence theorem can be used here just like it can be used in other circumstances with which you are familiar. Just think of $T^{\alpha i}$ as four different 3-vectors, one for each value of α:

$$\oint_S T^{\alpha i} dA^i = \int_V \frac{\partial T^{\alpha i}}{\partial x^i} dV \ . \tag{10.29}$$

Putting together our two formulations of Δp^α yields

$$\Delta t \int_V \left[\frac{\partial T^{\alpha 0}}{\partial x^0} + \frac{\partial T^{\alpha i}}{\partial x^i}\right] dV = 0 \ . \tag{10.30}$$

This gives a continuity equation for the stress-energy tensor:

$$\partial_\beta T^{\alpha\beta} = 0 \ . \tag{10.31}$$

This equation expresses both conservation of energy and conservation of momentum for the material whose stress-energy tensor is $T^{\alpha\beta}$.