Massachusetts Institute of Technology
Department of Physics
8.033 Fall 2024

Lecture 14
Prelude to gravity: More on the uniformly accelerated observer

14.1 The trajectory of an accelerated observer

In this lecture, as a prelude to discussing certain aspects of gravity, we will look at how things appear to observers who are accelerating. *A word of caution*: some of the calculations we do here are a touch tricky. Certain details require us to develop things beyond the level that is part of the normal core of 8.033; those details are developed toward the end of this set of lecture notes. Do not worry if you cannot follow every calculational detail in this set of notes. We emphasize the core important pieces of this analysis where appropriate, and lay out why they are important for where we are going next. A few of the sections we present below are significantly more complicated than what you are expected to follow; those sections can be skipped, though interested students who wish to discuss them further are welcome to do so.

We begin by examining the trajectory of a single observer who feels a constant acceleration $\mathbf{g} = g\mathbf{e}_x$ in their own momentarily comoving rest frame (MCRF). In the previous lecture, we found that such an observer has a 4-velocity whose components are

$$c\frac{dt}{d\tau} = u^t = c\cosh(g\tau/c) , \tag{14.1}$$

$$\frac{dx}{d\tau} = u^x = c\sinh(g\tau/c) . \tag{14.2}$$

(To simplify the analysis which follows, which is fairly dense, we take the observer to be at rest in the y and z directions.) Integrating up these solutions, we find the ct and x coordinates describing a uniformly accelerated observer, parameterized by that observer's own proper time:

$$ct = ct_0 + \frac{c^2}{g}\sinh(g\tau/c) , \tag{14.3}$$

$$x = x_0 + \frac{c^2}{g}\left(\cosh(g\tau/c) - 1\right) . \tag{14.4}$$

We've chosen constants of integration so that $t = t_0$ and $x = x_0$ at $\tau = 0$. The blue curve in Figure 1 shows what this motion looks like, choosing $x_0 = c^2/g$ and $t_0 = 0$.

At any moment as the accelerating observer moves along their worldline, we can find their 3-velocity: it is entirely in the x direction, and has magnitude

$$v^x = c\,u^x/u^t = c\,\tanh(g\tau/c) . \tag{14.5}$$

(Notice that the accelerating observer's *rapidity*, which you used on problem sets 2 and 3, increases linearly as a function of that observer's proper time.) Knowing this v^x lets us work out the Lorentz transformation that takes us from inertial coordinates (ct, x) that are at rest with respect to the observer's initial condition to the coordinates $(c\bar{t}, \bar{x})$ corresponding to their MCRF. Figure 1 shows the motion of the accelerating observer according to an inertial observer who is initially at rest with respect to the accelerating observer, along with several examples of constant $\bar{t}$ surfaces in the (ct, x) coordinates for this observer's MCRF at different moments along their worldline. At $\tau = 0$, the constant $\bar{t}$ surface in the MCRF coincides with the $t = 0$ surface in the inertial coordinates. As proper time grows along the worldline, these surfaces grow steeper as the observer moves faster with respect to their original rest frame.

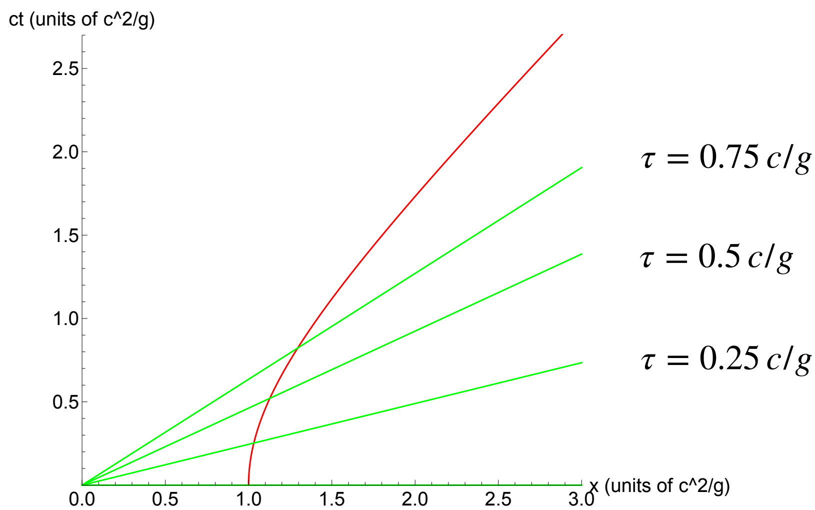

Figure 1: Worldline of an accelerating observer that starts at $x = c^2/g$ (red curve), and three examples of the constant $\bar{t}$ coordinates of that observer's MCRF at different moments along the worldline. The MCRF time $\bar{t}$ coincides with the observer's proper time τ where it crosses the worldline. Notice both axes are in units of c^2/g.

14.2 Comparing the worldlines of *two* accelerated observers: Breakdown of clock synchronization

Now imagine that there are two accelerated observers. Both are at rest with respect to the "unbarred" frame at $t = 0$, and both feel constant acceleration g. One (which will call the "trailing" observer) begins at $x_0 = c^2/g$; the other (the "leading" observer) begins at $x_0 = c^2/g + L$. Let the time as measured on the trailing observer's clock be $\bar{t}$; let the time as measured on the leading observer's clock be $\bar{\bar{t}}$. These times will also be used to describe time in the MCRFs along the accelerating observers' worldlines.

The clocks on these observers start out in agreement, and coincide with the initial inertial frame: when $t = 0$, $\bar{t} = \bar{\bar{t}} = 0$. However, it is not hard to see that as the two observers move along their worldlines, their clocks quickly fall out of agreement. Figure 2 illustrates the situation: once they begin moving, each observer's constant time surface tips over, in accordance with the Lorentz transformation that takes us from the inertial frame into their MCRF. However, they each tip about a different "pivot point," anchored to their own worldline. For a given value of proper time along the worldlines, the constant time $\bar{\bar{t}}$ surface used by the leading observer (whose worldline is illustrated by the orange curve in Fig. 2) always appears in the past of the constant time $\bar{t}$ surface used by the trailing observer (whose worldline is illustrated by the red curve).

This means that, when the leading observer measures time $\bar{\bar{t}} = 0.5\, c/g$ (for example), this is simultaneous with the trailing clock reading some value $\bar{t} < 0.5\, c/g$. The trailing observer agrees with this assessment: when they measure $\bar{t} = 0.5\, c/g$, this is simultaneous with the leading clock reading some value $\bar{\bar{t}} > 0.5\, c/g$. Both observers agree that the leading clock runs faster than the trailing clock.

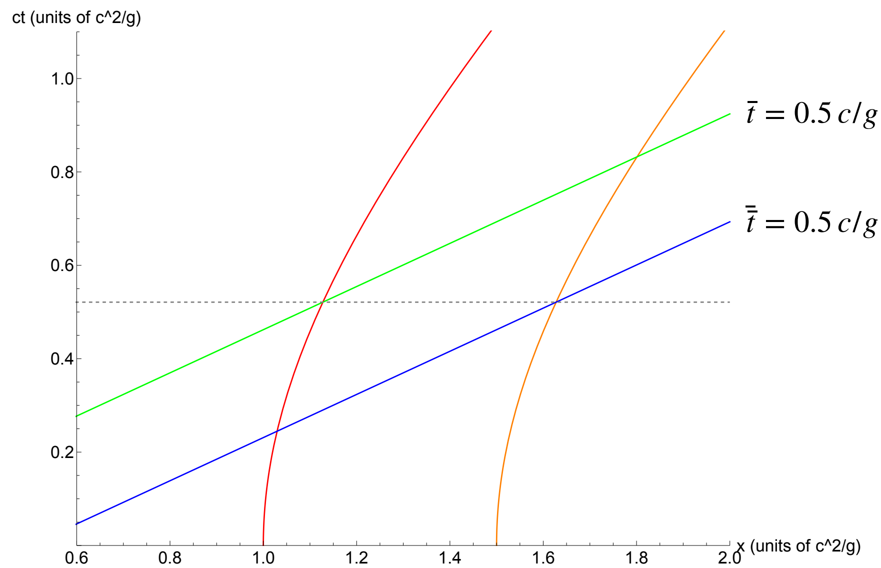

Figure 2: Worldline of two accelerating observers. Both feel acceleration g, and both are initially at rest in the coordinates (ct, x). The trailing observer (red curve) uses the time coordinate $\bar{t}$; the leading observer (orange curve) uses the time coordinate $\bar{\bar{t}}$. We show two surfaces of constant time according to the MCRF of the two observers. The green line shows the $\bar{t} = 0.5\, c/g$ surface; this corresponds to the MCRF of the trailing (red) when $\tau_{\text{trailing}} = 0.5\, c/g$. The blue line shows the $\bar{\bar{t}} = 0.5\, c/g$ surface; it corresponds to the MCRF of the leading (orange) worldline when $\tau_{\text{leading}} = 0.5\, c/g$. The constant $\bar{t}$ surface intersects the red worldline at $t \simeq 0.521\, c/g$; the constant $\bar{\bar{t}}$ surface intersects the orange worldline at the same value of t. (The dashed gray line is a constant at $t = 0.521\, c/g$.) These surfaces tell us that **the leading clock** (i.e., the clock of the observer at larger x) **runs fast** compared to the trailing clock. Surfaces of constant $\bar{\bar{t}}$ are consistently in the past of surface of constant $\bar{t}$, meaning that a particular value of $\bar{\bar{t}}$ has already happened by the time $\bar{t}$ reaches that value. Both observers agree that the trailing clock is slower than the leading clock.

By borrowing some results from the discussion below of "Rindler coordinates," we can compute the precise amount by which the leading clock runs ahead of the trailing clock, at least when the speeds of the two accelerated observers in the inertial coordinate frame is small compared to light. Let us write down the worldlines of the trailing and leading observers as seen in the inertial coordinate system:

$$ct_T = \frac{c^2}{g}\sinh(g\bar{t}/c)\ , \qquad x_T = \frac{c^2}{g}\cosh(g\bar{t}/c)\ ; \tag{14.6}$$

$$ct_L = \frac{c^2}{g}\sinh(g\bar{\bar{t}}/c)\ , \qquad x_L = \frac{c^2}{g}\cosh(g\bar{\bar{t}}/c) + L\ . \tag{14.7}$$

Let us also write down how one represents a single slice of $\bar{t}$ = constant in the MCRF of the trailing observer:

$$ct_{\mathrm{MCRF},T} = x\ \tanh(g\bar{t}/c)\ . \tag{14.8}$$

This relationship is worked out in the detailed discussion and derivation of Rindler coordinates, which is developed in the more advanced material presented below.

The question we'd like to answer is: What is the value of $\bar{\bar{t}}$ when the time on the constant $\bar{t}$ slice crosses the worldline of the leading observer — in other words, what is $\bar{\bar{t}}$ when $ct_{\mathrm{MCRF},T} = ct_L$? Plugging in the various definitions yields the equation we must solve:

$$ct_{\mathrm{MCRF},T}\Big|_{x=x_L} = ct_L\ , \tag{14.9}$$

which means

$$\left[\frac{c^2}{g}\cosh(g\bar{\bar{t}}/c) + L\right]\tanh(g\bar{t}/c) = \frac{c^2}{g}\sinh(g\bar{\bar{t}}/c) \tag{14.10}$$

or

$$\left[\cosh(g\bar{\bar{t}}/c) + \frac{gL}{c^2}\right]\tanh(g\bar{t}/c) = \sinh(g\bar{\bar{t}}/c)\ . \tag{14.11}$$

We now need to solve Eq. (14.11) for $\bar{\bar{t}}$ as a function of $\bar{t}$. Remarkably, this isn't so hard to do, as long as a certain approximation holds. Begin by putting all of the terms that depend on $\bar{t}$ on the left-hand side, and all of the terms that depend on $\bar{\bar{t}}$ on the right:

$$\begin{aligned}\tanh(g\bar{t}/c) &= \frac{\sinh(g\bar{\bar{t}}/c)}{\cosh(g\bar{\bar{t}}/c) + gL/c^2}\\ &\simeq \tanh(g\bar{\bar{t}}/c)\left[1 - \frac{gL}{c^2\cosh(g\bar{\bar{t}}/c)}\right]\ .\end{aligned} \tag{14.12}$$

The approximation introduced here is accurate as long as $gL/c^2 \ll \cosh(g\bar{\bar{t}}/c)$; recalling that c^2/g is roughly 1 light-year for an acceleration $g = 9.8\,\mathrm{m/s}^2$, this is clearly reasonable as long as L is anything much smaller than a light year.. Taking the arc-hyperbolic tangent of both sides, and using the result[1]

$$\mathrm{arctanh}\left[\tanh(x)(1-\epsilon)\right] \simeq x - \cosh(x)\sinh(x)\epsilon\ , \tag{14.13}$$

we find

$$g\bar{t}/c = g\bar{\bar{t}}/c - \frac{gL}{c^2}\sinh(g\bar{\bar{t}}/c)\ . \tag{14.14}$$

For general values of $\bar{\bar{t}}$, this isn't too easy to work with. However, if we confine ourselves to $g\bar{\bar{t}}/c \ll 1$, then this simplifies very nicely: using $\sinh(x) \simeq x$ for $x \ll 1$, Eq. (14.14) becomes

$$\bar{t} = \bar{\bar{t}}\left(1 - \frac{gL}{c^2}\right)\ . \tag{14.15}$$

[1]Figuring out things like this is a good use for tools like Mathematica.

The leading clock ticks at a faster rate than the trailing clock:

$$\frac{\bar{\bar{t}} - \bar{t}}{\bar{\bar{t}}} = \frac{gL}{c^2} \; . \tag{14.16}$$

Remember this nice, clean result! We will soon see a similar form when examining a different quantity, and rediscover this result in another context in a few lectures.

14.3 Light measured by the two accelerated observers

A related calculation compares the properties of light as measured by the two observers. This is particularly important because light plays such a critical role in relativity, since we often exploit the fact that its speed is c in all reference frames. Let's imagine that a beam of light travels in the $+x$ direction. It first intersects the trailing observer's worldline, then continues and later intersects the leading observer's worldline. The question we want to know is: What is the energy that the two observers measure for this light?

We will do all of these calculations in the inertial frame, which provides a convenient "stage" for us to formulate the quantities that we need for this analysis. We will also use the fact that, given something with 4-momentum $\vec{p}$, an observer whose 4-velocity is $\vec{u}$ measures it to have energy $E = -\vec{p} \cdot \vec{u}$.

Begin by writing the components of the light's 4-momentum in the inertial frame as

$$p^t = h\nu/c \; , \qquad p^x = h\nu/c \; . \tag{14.17}$$

(The y and z components of the light's 4-momentum are zero.) Let us say that this light crosses the worldline of the trailing observer when that observer's clock reads $\bar{t}_{\rm beam}$. Their 4-velocity at that time has components in the inertial frame

$$u^t_T = c\,\cosh(g\bar{t}_{\rm beam}/c) \; , u^x = c\,\sinh(g\bar{t}_{\rm beam}/c) \; . \tag{14.18}$$

The energy that the trailing observer measures for the light is then given by

$$E_T = -\vec{p} \cdot \vec{u}_T \tag{14.19}$$

$$= h\nu\cosh(g\bar{t}_{\rm beam}/c) - h\nu\sinh(g\bar{t}_{\rm beam}/c) \tag{14.20}$$

$$= h\nu\cosh(g\bar{t}_{\rm beam}/c)\,(1 - \tanh(g\bar{t}_{\rm beam}/c)) \; . \tag{14.21}$$

This can be simplified a bit more using a few hyperbolic function identities:

$$\cosh(x) = \frac{1}{\sqrt{\mathrm{sech}^2(x)}} = \frac{1}{\sqrt{1 - \tanh^2(x)}} \; . \tag{14.22}$$

Using this, we see that the energy measured by the trailing observer is

$$E_T = h\nu\sqrt{\frac{1 - \tanh(g\bar{t}_{\rm beam}/c)}{1 + \tanh(g\bar{t}_{\rm beam}/c)}} \; . \tag{14.23}$$

Notice that this is exactly the Doppler shift that one expects for an observer who is moving away from a light source with 3-speed $v = c\tanh(g\bar{t}_{\rm beam}/c)$.

The light continues to move in the $+x$ direction, and crosses the worldline of the leading observer when their clock reads $\bar{\bar{t}}_{\rm beam}$. By a similar calculation, the energy that the leading observer measures is

$$E_L = h\nu\sqrt{\frac{1-\tanh(g\bar{\bar{t}}_{\rm beam}/c)}{1+\tanh(g\bar{\bar{t}}_{\rm beam}/c)}}\ , \tag{14.24}$$

which is likewise just the Doppler-shifted energy for a speed $v = c\tanh(g\bar{\bar{t}}_{\rm beam}/c)$.

We'd like to compare E_T to E_L. To do so, we must relate the time $\bar{\bar{t}}_{\rm beam}$ at which the light beam crosses the leading observer's worldline to the time $\bar{t}_{\rm beam}$ at which the beam crosses the trailing observer's worldline. We do this by using our results describing time in the inertial frame to the times along the worldline.

The inertial-frame time at which the light crosses the trailing observer's worldline is

$$t_T = \frac{c}{g}\sinh(g\bar{t}_{\rm beam}/c)\ ; \tag{14.25}$$

the inertial-frame time at which it crosses the leading observer's worldline is

$$t_L = \frac{c}{g}\sinh(g\bar{\bar{t}}_{\rm beam}/c)\ . \tag{14.26}$$

However, we also know that, in the inertial frame, the light moves a distance of L in going from the trailing observer to the leading observer, plus the additional distance that the leading observer covers while the light is in transit:

$$\begin{aligned} t_L &= t_T + \frac{L}{c} + \int_{t_T}^{t_L} \frac{dx}{dt}dt \\ &= t_T + \frac{L}{c} + c\int_{t_T}^{t_L} \tanh(g\bar{\bar{t}}/c)\, dt\ . \end{aligned} \tag{14.27}$$

The integral on the last line accounts for the distance that the leading observer moves as the light is in transit. As written, it is not a very nice integral: we do the integral with respect to the inertial-frame time, but the function we are integrating is parameterized using time $\bar{\bar{t}}$ along that observer's worldline. So we, need to convert: using Eq. (14.3) (with $t_0 = 0$, and with $\tau = \bar{\bar{t}}$), we have

$$\bar{\bar{t}} = \frac{c}{g}\text{arcsinh}(gt/c)\ , \tag{14.28}$$

and the argument of the integral becomes

$$\begin{aligned} \tanh(g\bar{\bar{t}}/c) &= \tanh(\text{arcsinh}(gt/c)) \\ &= \frac{(gt/c)}{\sqrt{1+(gt/c)^2}}\ . \end{aligned} \tag{14.29}$$

It's kind of miraculous that this result cleans up so nicely. We can now easily do the integral and relate t_L to t_T:

$$t_L = t_T + \frac{L}{c} + \frac{c}{g}\left(\sqrt{1+(gt_L/c)^2} - \sqrt{1+(gt_T/c)^2}\right)\ . \tag{14.30}$$

We now have all the information we need, in principle, to see how the energy of the light changes as it goes from the trailing observer to the leading one:

1. Solve Eq. (14.30) to find t_L as a function of t_T.

2. Using this solution plus Eqs. (14.24) and (14.26), compute the energy measured by the leading observer as a function of t_T.

3. Using Eq. (14.25) and (14.23), compute the energy measured by the trailing observer as a function of t_T.

Unfortunately, this "recipe" involves a multitude of hyperbolic functions and does not yield a nice closed form answer. To get something tractable, let's assume that gt/c, $g\bar{t}/c$, and $g\bar{\bar{t}}/c$ are all much smaller than 1, and use the limiting forms

$$\cosh(x) \simeq 1 \;, \qquad \sinh(x) \simeq x \;, \qquad \tanh(x) \simeq x \qquad \text{when } x \ll 1 \;. \tag{14.31}$$

Doing so, we find

$$t_L \simeq t_T + \frac{L}{c} \;, \tag{14.32}$$

$$t_T \simeq \bar{t}_{\rm beam} \;, \quad t_L \simeq \bar{\bar{t}}_{\rm beam} \;, \tag{14.33}$$

$$E_T \simeq h\nu\sqrt{\frac{1-(g\bar{t}_{\rm beam})/c}{1+(g\bar{t}_{\rm beam})/c}} \simeq h\nu\left(1 - g\bar{t}_{\rm beam}/c\right) \;, \tag{14.34}$$

$$E_L \simeq h\nu\sqrt{\frac{1-(g\bar{\bar{t}}_{\rm beam})/c}{1+(g\bar{\bar{t}}_{\rm beam})/c}} \simeq h\nu\left(1 - g\bar{\bar{t}}_{\rm beam}/c\right) \;. \tag{14.35}$$

Putting all these together, we see that

$$\Delta E \equiv E_T - E_L \simeq h\nu\left(\frac{gL}{c^2}\right) \;. \tag{14.36}$$

The light's energy as measured by the leading observer is lower than the energy measured by the trailing observer, by a fractional amount that precisely matches the rate at which their clock ticks faster than the trailing observer's clock.

14.4 Wrapup: Key things to take away

The calculations that went into the above discussion were somewhat dense, so this is a good point to pause and assess the key lessons that we should take away from it. In particular, we want to emphasize aspects of what is observed by a pair of observers who share the same acceleration $\mathbf{g}$, but are spatially separated by a distance L.

- Even if the observers start out with their clocks perfectly synchronized, they will fall out of synchrony as time passes, with the leading clock running fast by a factor gL/c^2.

- If light is exchanged between the two observers, they will disagree on its energy. The leading observer measures it to have a lower energy (i.e., they see the light as being somewhat redder), by a factor gL/c^2.

As our analysis showed, the numerical factor gL/c^2 that emerges from these analyses is an approximate one, but works well as long as $g(\text{time})/c$ is small for all the versions of "time" under consideration. Bear in mind that $c/g \simeq 1\,\text{year}$ if $g = 10\,\text{m/s}^2$; this gives a sense of the time and lengthscales involved before these approximations start to break down.

14.5 Rindler coordinates (CAUTION: somewhat advanced material)

Parts of the discussion in the preceding few sections rely on more advanced material which we present here. We recommend that you read these notes, but you should not be worried if you do not follow every detail of this discussion. The nature of the Rindler coordinates, Eqs. (14.37)–(14.40), and the subsection labeled "Features of the Rindler representation" are particularly worth your attention.

In almost all of our discussion so far, we have used coordinates (t, x, y, z) or (ct, x, y, z) that are particularly well suited for describing inertial observers. Indeed, such coordinates are often called *inertial coordinates*: they are ones for which there exists some set of observers who sit at constant (x, y, z). In such a frame, the observers are only "moving" in time. There are also many observers who move with constant velocity. The worldlines of the constant velocity observers are lines in these coordinates, taking the form $x = x_0 + v^x t$, and similarly for their motion in y and z.

Even when we discussed accelerating observers, we presented their motion as seen by some inertial observer who sees the accelerating observer zoom past. You might wonder — how does the accelerating observer describe spacetime? Do we learn anything useful by developing coordinates that are "adapted" to the reference frame of the acelerating observer? To do this, one could imagine performing Lorentz transformations that flip between a particular inertial frame (e.g., the frame used to draw the time axes in Fig. 1) and the accelerating observer's MCRF. However, the relative velocity of the MCRF and any given inertial observer is continually changing. The Lorentz transformations that enact this "flipping back and forth" thus must continually evolve, which limits their usefulness for us.

A coordinate system which nicely describes an accelerating observer in fact can be written down. These coordinates (named *Rindler coordinates*, in honor of Wolfgang Rindler who did much to explore their properties and applications) are described and explored in this section. The following section derives Rindler coordinates; that section should be considered even more advanced than this one. Students should feel free to ignore it altogether.

Let us choose the initial condition of the accelerated observer's trajectory so that $t_0 = 0$ and $x_0 = c^2/g$ in Eqs. (14.3) and (14.4). Then, as we derive in detail in the following section, the accelerated observer uses coordinates $(c\bar{t}, \bar{x}, \bar{y}, \bar{z})$ to describe spacetime. These new coordinates are related to the original "inertial" coordinates (ct, x, y, z) according to

$$ct = \bar{x}\sinh(g\bar{t}/c)\ , \tag{14.37}$$

$$x = \bar{x}\cosh(g\bar{t}/c)\ , \tag{14.38}$$

$$y = \bar{y}\ , \tag{14.39}$$

$$z = \bar{z}\ . \tag{14.40}$$

In the barred coordinate system, the accelerated observer is at constant spatial coordinate $(\bar{x}, \bar{y}, \bar{z}) = (c^2/g, 0, 0)$; the barred time coordinate $\bar{t}$ is exactly the same as the proper time τ that this observer measures. Notice that this solution agrees with Eqs. (14.3) and (14.4) when $\bar{x} = c^2/g$. Equations (14.37)–(14.40) define the Rindler coordinates. (Notice also that Eq. (14.37) is what we used to define the constant time surfaces of the MCRF as shown in Fig. 1 and in the associated discussion.)

Figure 3 illustrates how the $(c\bar{t}, \bar{x})$ coordinates used by an accelerating observer appear in the reference frame of an unaccelerated observer. The red curve illustrates the worldline of the observer who starts at $x = \bar{x} = c^2/g$. The green lines represent surfaces of constant $\bar{t}$; the blue hyperbolic curves represent trajectories of constant $\bar{x}$. Those trajectories are chosen by requiring that $\bar{x} = x$ when $t = \bar{t} = 0$, and by demanding that the unit vector along $\bar{x}$ be spacetime orthogonal to the unit vector along $\bar{t}$. Notice that each constant $\bar{x}$ coordinate can itself be regarded as an accelerated observer; as we discuss in the next section, it can be shown that the observer at constant $\bar{x}$ feels an acceleration $\mathbf{a} = (c^2/\bar{x})\mathbf{e}_x$.

We also include in this figure the trajectory of a light ray that is emitted from the origin; we discuss some interesting features of this coordinate system's behavior with respect to this light ray below.

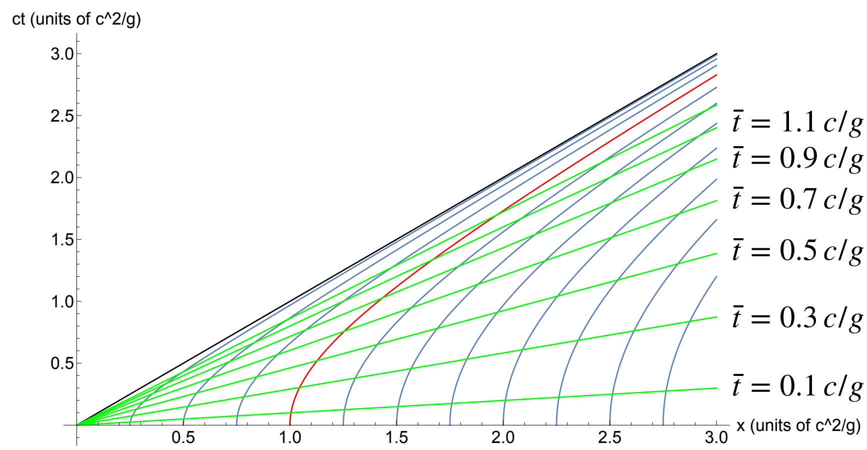

Figure 3: An illustration of Rindler coordinates. The red curve is the worldline of an accelerating observer who starts at $x = c^2/g$ and experiences constant acceleration g. The green lines are surface of constant $\bar{t}$, which coincides at that observer's location with their own proper time; the blue curves are trajectories of constant $\bar{x}$, chosen to coincide with the unaccelerated frame's x when $t = \bar{t} = 0$. A heavy black line $ct = x$ illustrates a light ray that is emitted from the origin and moves to the right. Notice both axes are in units of c^2/g.

14.5.1 Features of the Rindler representation

There are two features of the Rindler representation to which we would like to particularly call your attention.

- **A new form for the metric:** By now, we know very well that

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 \equiv \eta_{\alpha\beta}dx^\alpha\, dx^\beta\ . \qquad (14.41)$$

  The invariance of this interval is what led us to the metric used in inertial coordinates, $\eta_{\alpha\beta} = \text{diag}(-1, 1, 1, 1)$.

Let's look at this in our new coordinates. Considering Eqs. (14.37)–(14.38), we see

$$c\,dt = (d\bar{x})\sinh(g\bar{t}/c) + \frac{g\bar{x}}{c^2}(c\,d\bar{t})\cosh(g\bar{t}/c)\ , \tag{14.42}$$

$$dx = (d\bar{x})\cosh(g\bar{t}/c) + \frac{g\bar{x}}{c^2}(c\,d\bar{t})\sinh(g\bar{t}/c)\ , \tag{14.43}$$

plus $dy = d\bar{y}$, $dz = d\bar{z}$. This tells us that

$$\begin{aligned} ds^2 &= -\left[(d\bar{x})\sinh(g\bar{t}/c) + \left(\frac{g\bar{x}}{c^2}\right)(c\,d\bar{t})\cosh(g\bar{t}/c)\right]^2 \\ &\quad + \left[(d\bar{x})\cosh(g\bar{t}/c) + \left(\frac{g\bar{x}}{c^2}\right)(c\,d\bar{t})\sinh(g\bar{t}/c)\right]^2 + d\bar{y}^2 + d\bar{z}^2 \\ &= -\left(\frac{g\bar{x}}{c^2}\right)^2 c^2 d\bar{t}^2 + d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2\ . \end{aligned} \tag{14.44}$$

(We used $\cosh^2(g\bar{t}/c) - \sinh^2(g\bar{t}/c) = 1$.) Notice that the metric *is not a constant* in this representation. Because we reserve the symbol $\eta_{\alpha\beta}$ for diag$(-1, 1, 1, 1)$, we now use $g_{\alpha\beta}$ to denote the metric. In particular, we now have

$$g_{\alpha\beta} \doteq \mathrm{diag}\left(-(g\bar{x}/c^2)^2, 1, 1, 1\right) \tag{14.45}$$

for the metric of spacetime in Rindler coordinates.

It's worth emphasizing that we *are still doing special relativity*; we have only changed coordinates. If you've been reading ahead or poking at references, you may have seen that in general relativity we get metrics in which the components are functions, and so you might worry that we've somehow "broken" special relativity. We haven't: in some coordinate systems, the components of the metric are functions and yet the metric still describes special relativity. This is an example of such a system.

- **A "horizon":** Notice in Fig. 3 that we have included a light ray that starts at the origin and travels in the $+x$ direction. On an upcoming problem set, you will compare the motion of the accelerated observer to the motion of this light ray, and show that the light ray *never* crosses this observer's trajectory. The light ray asymptotically approaches the accelerated observer's trajectory as $\bar{t} \to \infty$, but they never cross. In fact, the light ray never crosses *any* of the constant $\bar{x}$ trajectories.

  Because information can travel no faster than light, this means that there is a region of spacetime that *cannot communicate with the accelerated observer.* No signal sent by an observer to the "left" of that light ray can reach the accelerated observer. We say that there is a *horizon* separating the events which can communicate with the accelerated observer from those events which cannot so communicate.

  We will come back to the notion of horizons later in this course. Take this as a preview of some of the interesting features that we will begin to find as we start investigating certain spacetimes.

14.6 Derivation of Rindler coordinates (CAUTION: advanced material)

The discussion in this section is significantly more advanced than is expected for 8.033 students. It is included in order to provide a complete explanation of where the Rindler coordinates come from, as well as for the benefit of any students who are interested in diving somewhat deeper into this material; it will *not* be discussed in detail during lecture.

We now define coordinates $c\bar{t}$, $\bar{x}$ which the accelerating observer uses to describe spacetime. (Since the acceleration is along x, we simply put $\bar{y} = y$ and $\bar{z} = z$ and are then done with those two coordinates.) We take the accelerating observer's coordinates to be $t = 0$, $x = c^2/g$ when $\tau = 0$, and we use the symbols T, X to define the accelerating observer's trajectory as measured by the observer who is at rest with respect to the accelerating observer at $\tau = 0$. The motion of this observer is thus

$$cT(\tau) = \frac{c^2}{g}\sinh(g\tau/c) , \qquad X(\tau) = \frac{c^2}{g}\cosh(g\tau/c) . \tag{14.46}$$

For the accelerated observer, their own proper time τ makes a natural clock. Given this, it is natural that the accelerated observer chooses the time coordinate to be $\bar{t} = \tau$ along their own worldline.

Can we use this coordinate $\bar{t}$ *away* from the observer's worldline? In other words, can the accelerating observer use $\bar{t}$ to label events elsewhere in spacetime, away from their own worldline? Yes, by the following procedure:

- First define unit vectors that point along the directions $\bar{t}$ and $\bar{x}$. Making such a unit vector for $\bar{t}$ is not hard: in the accelerating observer's MCRF, their 4-velocity has components $u^\alpha \doteq (c, 0, 0, 0)$. A natural choice for $\vec{e}_{\bar{t}}$ is thus parallel to this observer's 4-velocity, so we put

  $$\vec{e}_{\bar{t}} = \frac{1}{c}\vec{u} = \cosh(g\bar{t}/c)\vec{e}_t + \sinh(g\bar{t}/c)\vec{e}_x . \tag{14.47}$$

  We then define $\vec{e}_{\bar{x}}$ by requiring that it be orthogonal to $\vec{e}_{\bar{t}}$ (and also that it have no components along $\bar{y}$ and $\bar{z}$):

  $$\vec{e}_{\bar{x}} = \sinh(g\bar{t}/c)\vec{e}_t + \cosh(g\bar{t}/c)\vec{e}_x . \tag{14.48}$$

- With $\vec{e}_{\bar{x}}$ defined, now consider a "surface" of constant $\bar{t}$ (i.e., a set of events in which all the time coordinates $\bar{t}$ are the same). Such a surface must lie on a line that is parallel $\vec{e}_{\bar{x}}$, meaning that it is a line whose slope m is given by

  $$m = \frac{e^t_{\bar{x}}}{e^x_{\bar{x}}} = \tanh(g\bar{t}/c) . \tag{14.49}$$

  We further require that this line intersect the worldline of the accelerating observer: The line must have the slope m defined by Eq. (14.49), and pass through the point $[cT(\bar{t}), X(\bar{t})]$. With a little algebra we see that this line is given by

  $$ct = x\tanh(g\bar{t}/c) . \tag{14.50}$$

We've now learned how to draw surfaces of constant $\bar{t}$ in the inertial (ct, x) coordinate frame. How do we draw a surface of constant $\bar{x}$? Such a surface must lie parallel to the timelike vector $\vec{e}_{\bar{t}}$ given in Eq. (14.47). This vector is continually changing in slope as $\bar{t}$ changes; in the inertial frame, it has slope

$$\frac{dx}{dt} = c\tanh(g\bar{t}/c) \; . \tag{14.51}$$

We have already deduced that t and $\bar{t}$ are related by Eq. (14.50). Combining these results, we see that

$$\frac{dx}{dt} = c^2 \frac{t}{x} \; . \tag{14.52}$$

We integrate this up, applying an initial condition that the coordinates of the accelerated observer match those of the inertial frame at $t = \bar{t} = 0$:

$$\int_{\bar{x}}^{x} x\, dx = c^2 \int_0^t t\, dt \tag{14.53}$$

or

$$x^2 - \bar{x}^2 = c^2 t^2 \; . \tag{14.54}$$

This tells us that surfaces of constant $\bar{x}$ are given by hyperbolae in the (ct, x) plane which satisfy

$$\bar{x}^2 = x^2 - (ct)^2 \; . \tag{14.55}$$

We'd like to massage Eqs. (14.50) and (14.55) a bit more to really isolate how $(c\bar{t}, \bar{x})$ appear in the inertial frame. Notice that Eq. (14.55) is solved by any pair of functions of the form

$$x = \bar{x}\cosh(\alpha) \; , \qquad ct = \bar{x}\sinh(\alpha) \; . \tag{14.56}$$

Applying this to Eq. (14.50), we see that we must have $\alpha = g\bar{t}/c$. We thus at last have the complete mapping of the accelerated observer's reference frame into the inertial coordinate system:

$$ct = \bar{x}\sinh(g\bar{t}/c) \; , \qquad x = \bar{x}\cosh(g\bar{t}/c) \; , \qquad y = \bar{y} \; , \qquad z = \bar{z} \; . \tag{14.57}$$

One final detail: it was noted earlier in these notes that an observer at constant $\bar{x}$ is itself an accelerated observer. This is hopefully intuitively obvious from the shape of the constant $\bar{x}$ surfaces in Fig. 3 (if they were not accelerated, they would not curve). What acceleration does this observer feel? This is most easily calculated by computing the 3-acceleration of this observer at $t = \bar{t} = 0$. Because at this moment all of the constant $\bar{x}$ observers happen to be momentarily at rest, all of these observers have 4-velocity with components $(c, 0, 0, 0)$ and 4-acceleration $(0, a^x, 0, 0)$ in this frame, where $a^x = d^2x/dt^2$ at $t = 0$.

Let's compute this:

$$\begin{aligned}
a^x &= \left.\frac{d^2x}{dt^2}\right|_{t=\bar{t}=0} \\
&= \left[\frac{d^2x}{d\bar{t}^2}\left(\frac{dt}{d\bar{t}}\right)^{-2}\right]_{t=\bar{t}=0} \\
&= \left[\left(\frac{g^2}{c^2}\bar{x}\cosh(g\bar{t}/c)\right)\left(\frac{g}{c}\frac{\bar{x}}{c}\cosh(g\bar{t}/c)\right)^{-2}\right]_{t=\bar{t}=0} \\
&= \frac{c^2}{\bar{x}} \; .
\end{aligned} \tag{14.58}$$

So the observer at $\bar{x} = c^2/g$ feels an acceleration of precisely g; those at larger $\bar{x}$ feels less acceleration, and those at smaller $\bar{x}$ feel more (with the acceleration diverging as $\bar{x} \to 0$).