MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF PHYSICS
8.033 FALL 2021

LECTURE 15
INTRODUCTION TO GRAVITY

15.1 The road ahead

In this set of lecture notes, we are going to begin to examine how to incorporate gravity into relativity. We will be concerned with two major questions:

- First, how is "gravity made," broadly speaking? In other words, given some body that generates a gravitational field in Newtonian physics, how do we describe that body's gravity using relativistic physics?

- Second, how do we describe a body's motion under the influence of gravity? How does what we think of as the "gravitational force" act in relativistic physics?

You might imagine that, given all we have done so far, addressing these points shouldn't be too difficult. After all, we reformulated both electric and magnetic forces and fields into nicely covariant relativistic language. How much harder can this be for gravity?

As we'll begin to see in the next section of these notes, gravity introduces complications that make describing it *substantially* more difficult. Indeed, going through all the details in great rigor is far beyond the scope of 8.033. We will content ourselves in this class with a more descriptive analysis, seeing how it is that the tricks we've learned so far don't work for gravity. We will then examine a high-level synopsis of how we proceed to answer the first of the two questions above. Going beyond that high-level synopsis takes roughly half the term of 8.962. Students who wish to pursue this subject further are encouraged to look into the department's new IAP offering, and to consider taking 8.962 at some point down the road.

Once we have this high-level synopsis of how gravity arises, it isn't beyond 8.033 to describe how that gravity acts on a body. Exploring how relativistic gravity acts and how it differs from Newtonian gravity will be a big part of what we do in the last few weeks of this term. To get there, we first need to establish some important principles.

15.2 The principle of maximum aging

Imagine that two bodies travel from event A, located at $x = 0$, $t = 0$ to event B located at $x = 1$ lightsecond, $t = 4$ seconds. One body moves there at constant velocity $\mathbf{v} = 0.25c\,\mathbf{e}_x$; we'll call this the "direct" path. The other body moves first to event C, located at $x = 0$, $t = 2$ seconds. It then moves off to event C at half the speed of light[1]. We illustrate this situation in Fig. 1.

[1] In reality, the body must accelerate for some interval to reach this speed. For this initial discussion, we idealize the interval over which the acceleration occurs to be so short that it is nearly instantaneous.

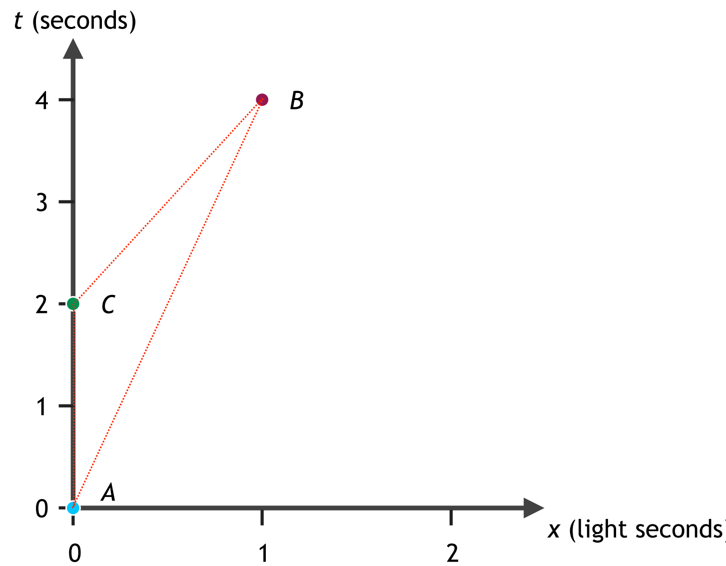


Figure 1: Two paths from event A to event B. The "direct" path goes from A to B at constant velocity; the indirect path goes from A to B via the event C. Note that different scales are being used for the x and t axes.

Question: On which path does the body age more, the direct one or the indirect one? We've already discussed a similar situation when talked about the twin paradox, but just to remind ourselves how this works let's step through the analysis. We are going to use the fact that along any timelike trajectory,

$$\Delta s^2 = -c^2 \Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 = -c^2 \Delta\tau^2 \; . \tag{15.1}$$

The last equality follows from the fact that $\Delta\tau$ is the time experienced in the body's own rest frame; in that frame, $\Delta x = \Delta y = \Delta z = 0$, since the body is at rest in its rest frame. Let's use this to compute how much $\Delta\tau$ the bodies experience along these two trajectories.

First, consider the direct trajectory:

$$\begin{aligned} \Delta\tau &= \left[(\Delta t)^2 - (\Delta x)^2/c^2\right]^{1/2} \\ &= \left[16\,\mathrm{sec}^2 - 1\,\mathrm{sec}^2\right]^{1/2} \\ &= \sqrt{15}\,\mathrm{seconds} \; . \end{aligned} \tag{15.2}$$

The body ages a total of $\sqrt{15} \simeq 3.87$ seconds on the direct trajectory.

Next, the indirect trajectory. We break this up into two pieces:

$$\Delta\tau_{A\to C} = 2\,\mathrm{seconds} \; . \tag{15.3}$$

$$\begin{aligned} \Delta\tau_{C\to B} &= \left[4\,\mathrm{sec}^2 - 1\,\mathrm{sec}^2\right]^{1/2} \\ &= \sqrt{3}\,\mathrm{seconds} \; . \end{aligned} \tag{15.4}$$

So on the indirect trajectory, the body ages a total of $2+\sqrt{3} \simeq 3.73$ seconds. *The body ages more on the unaccelerated trajectory.*

Without too much effort, we can find other trajectories in which the aging is less — *much* less, if we design the trajectory well. Consider, for example, the trajectory shown in Fig. 2:

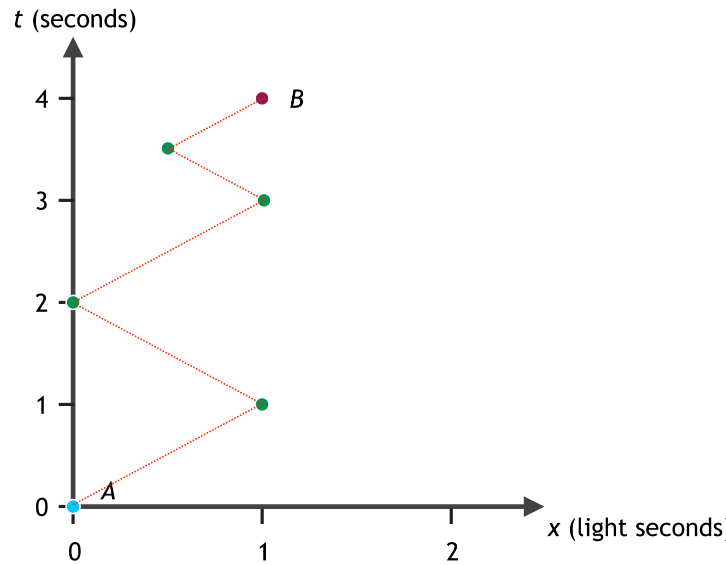


Figure 2: Yet another path from event A to event B. On this path, the body zig zags back and forth at nearly light speed until it finally reaches event B.

The path here zips back and forth at nearly light speed. As such, the body accumulates nearly *zero* proper time along each leg, and so it *does not age at all* in moving from event A to event B. It would appear that the effect of acceleration is to reduce the aging which a body experiences as it moves through spacetime.

In a few lectures, we are going to briefly discuss a topic called the *calculus of variations*. Some of you may have already learned about this topic; it is the key technique which underlies Lagrangian mechanics, for example. Using the calculus of variations, we can meaningfully pose the following question: "Given the infinite number of timelike trajectories in spacetime which connect events A and B, along which one does the body age *the most*? In other words, which trajectory through spacetime is the one which corresponds to maximum aging?"

The answer we will find is that the trajectory of maximum aging is indeed the unaccelerated trajectory. This will prove to be very, very useful for us. Picking out the "trajectory of maximum aging" to understand the motion of a body in special relativity is overkill; it is fine for understanding how this technique operates, but it isn't how you want to calculate a body's trajectory through special relativity's spacetime on an everyday basis. However, we will argue (using an important principle that Einstein introduced to understand how to incorporate gravity into relativity) that this technique is exactly what we need to compute a body's motion under gravity once we start making a relativistic theory of gravity.

For now, please file away in some mental storage drawer the idea that "no acceleration" means "maximum aging" as a body moves through spacetime. We will want to return to this point in several lectures.

15.3 Making Newton's gravity relativistic?

Long ago, Isaac Newton taught us that two masses feel a force that is proportional to their masses, inversely proportional to the square of the distance between them, directed along the line between the two masses, and attractive:

$$\mathbf{F}^{\mathrm{G}} = -G\frac{m_1 m_2}{r^2}\mathbf{e}_r \; . \tag{15.5}$$

This looks just like Coulomb's law, which tells us about the electric force between two charges:

$$\mathbf{F}^{\mathrm{E}} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\mathbf{e}_r \; . \tag{15.6}$$

The differences are that the electric force arises from charges q rather than masses m; the electric force can be attractive or repulsive, depending on the signs of q_1 and q_2 (note that masses are always positive); and the two forces have different "coupling constants" (G versus $1/4\pi\epsilon_0$). Given that we were able to promote the electric force to a relativistic analog without too much effort, with magnetic fields and forces arising as a consequence of this promotion, can we perhaps do the same thing for gravity?

Although this seems like a plausible course of action, it is important to recognize a major difference between the two force laws. The charges q_1 and q_2 which enter into the electric force are Lorentz invariants. Do all observers agree on the masses m_1 and m_2 which enter into the gravitational force?

The issue is yes **if** the Newtonian force only acts on *rest mass*. If that's the case, then it has an interesting consequence: gravity can have no effect on anything massless, such as light. It is also not hard to construct situations in which gravity acts a little oddly.

Consider a box of mass M. Inside this box are lumps of putty, each of rest mass m. If gravity only acts on rest mass, then on the surface of the Earth, this box will have weight

$$\mathbf{F}_w = (M + 2m)\mathbf{g} \; , \tag{15.7}$$

where

$$\mathbf{g} = \frac{GM_E}{R_E^2}\mathbf{e}_d \tag{15.8}$$

is the gravitational acceleration at the surface of the Earth. It depends on the Earth's mass M_E, its radius R_E, and points down, $\mathbf{e}_d$, from the surface toward the Earth's center.

Let's imagine that the lumps of putty are in fact moving toward each other at speeds very close to the speed of light: the box has a long axis oriented parallel to the Earth's surface (let's call this along the x direction); one lump of putty has $\mathbf{u} = u\mathbf{e}_x$, the other has $\mathbf{u} = -u\mathbf{e}_x$, with u close to c. Before the lumps of putty come into contact, the box's weight is given by Eq. (15.7). Afterwards, the box has weight

$$\mathbf{F}_w = (M + 2\gamma(u)m)\mathbf{g} \; . \tag{15.9}$$

If $u \sim c$, then $\gamma(u)$ can be huge; the weight of the box very suddenly increases.

Or, imagine that one lump of putty is made of matter, and the other of antimatter. After they collide, all of their rest mass is converted into radiation. If gravity only acts on rest mass, then the box now has weight

$$\mathbf{F}_w = M\mathbf{g} \tag{15.10}$$

after the collision; the weight very suddenly decreases. *If gravity only acts on rest mass, then an object's weight can very suddenly and discontinuously change.*

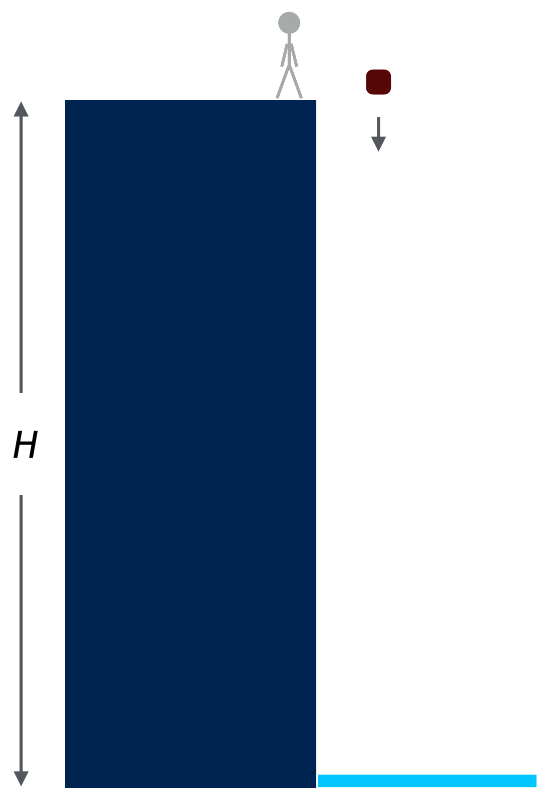
Let me emphasize that these thought experiments do *not* tell us that m_1 and m_2 are not rest masses. However, they make it clear that if they *are* rest mass, then we must be prepared for some potentially weird consequences. It is worth taking a few moments to think about alternatives.

What if gravity doesn't so much act on rest mass as it acts on *energy*? For instance, suppose that the m that appears in the Newtonian force law is really something like E/c^2. In the vast majority of situations that humanity encountered in its history, an object's kinetic energy is a very tiny fraction of its rest energy. As such, the difference between m and E/c^2 tended to be negligible. It is not unreasonable to imagine that gravity acts on energy, but that we inferred the force law (15.5) because of the overwhelming importance of rest energy at typical kinetic energy levels.

If gravity acts on *all* forms of energy, then it must act on light. Let's take a look at that possibility.

15.4 The action of gravity on light

Let's consider another thought experiment; this is a variation on one that was originally designed by Einstein.



- Imagine we stand on a tall building, and we drop a rock of mass m.
- After falling a distance H, the rock enters a device. At the moment it enters this device, it has energy

$$E = E_{\text{bottom}} = mc^2 + mgH \; . \tag{15.11}$$

- This device converts the rock into a single photon[2] of $E_{\rm bottom} = mc^2 + mgH = h\nu$ (be careful not to confuse the height H with Planck's constant h) and launches this photon back up to the top of the building.

- When the photon has climbed a distance H, we use yet another amazing device to convert the photon back into a rock. *What energy must this rock have?*

Imagine first that gravity does not act on light. If that were the case, then the rock would reappear next to us with energy $E = E_{\rm bottom} = mc^2 + mgH$ – it would either have a slightly larger rest mass, or else it would have some kinetic energy, and continue to climb. If we allowed it to come to a halt and then fall back down, on the next pass it would have energy $E = mc^2 + 2mgH$ at the end of this process. We can repeat this, giving the rock an extra mgH of energy on each go-round. *If gravity does not act on the light, then we can in principle make a device for creating unlimited amounts of energy*[3] *this way.*

Let's insist that energy be conserved: When the photon is converted back into a rock, it has an energy $E = E_{\rm top} = mc^2$. Because a photon's energy is related to its frequency, this tells us that the photon loses energy as it climbs out of the gravitation field: it is *redshifted* according to the rule

$$\frac{E_{\rm top}}{E_{\rm bottom}} = \frac{h\nu_{\rm top}}{h\nu_{\rm bottom}} = \frac{mc^2}{mc^2 + mgH} \ , \tag{15.12}$$

or, using $gH \ll c^2$,

$$\frac{\nu_{\rm top}}{\nu_{\rm bottom}} = 1 - \frac{gH}{c^2} \ . \tag{15.13}$$

Notice that this is *precisely* the same as the effect we found when we compared the energy of a photon that is measured by two accelerated observers; compare Sec. 14.3 of the previous set of lecture notes.

A few comments are worth making before moving on:

- The magnitude of this effect can be estimated by noting that, at the Earth's surface, $gH \simeq 100\,{\rm m^2/s^2}\,(H/10\,{\rm m})$, and by using $c^2 \simeq 9 \times 10^{16}{\rm m^2/s^2}$. This tells us that we expect a frequency change in the light of roughly 1 part in 10^{15} for every 10 meters of height change.

- A more general form of Eq. (15.13) is

$$\frac{\nu_{\rm top}}{\nu_{\rm bottom}} = 1 - \frac{\Delta\Phi_G}{c^2} \ , \tag{15.14}$$

  where $\Delta\Phi_G$ is the change in gravitational potential between the two measurement points.

[2] Alarm bells should be going off in your brain right now: Even allowing for the most amazing technology we can imagine, converting a single rock into a single photon would cause all sorts of problems with energy and momentum conservation. To address this, imagine dropping a rock and an "anti-rock" — a rock made of antimatter. The device can then create *two* photons; by mounting the device on the Earth, we can allow the Earth to recoil in such a way that both energy and momentum are conserved.

[3] The device used in this example is, by design, kind of silly. However, it is not hard to imagine making less silly variations on this. For example, by allowing matter and antimatter to fall in a gravitational field and then harvesting the light they produce upon annihilation, we could make any amount of energy we want, perhaps harvesting the energy by allowing those photons to heat up a bucket of water. The failure of gravity to act on light would be an on-ramp to building a perpetual motion machine.

We emphasize these points because this effect in fact is exactly what we measure. It was first done in 1959 using Mössbauer spectroscopy by Robert Pound and Glen Rebka, looking at the effect of gravity on gamma rays which produced by the decay of the isotope ^{57}Fe and then climbed 22.5 meters up a tower at Harvard's Jefferson Laboratory. This measurement is now done millions of times a second by a huge number of people around the world, as it is integral to the functioning of the Global Position System. Without correcting for this frequency shift, GPS accuracy would degrade at a rate of roughly 8 meters per minute.