Massachusetts Institute of Technology
Department of Physics
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Lecture 18
Some important solutions of the Einstein field equation; using those solutions

18.1 Final thoughts on the Einstein field equation

In the previous lecture, we discussed the generic framework and logic that led Einstein, after roughly a decade of learning the relevant mathematics and considering how to connect the pieces together, to the field equation of general relativity:

$$G^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu} \; . \tag{18.1}$$

The left-hand side of this equation (the "Einstein curvature tensor") can be regarded as a very complicated second-order differential operator acting on the metric of spacetime. It describes, after a little bit of massaging, the spacetime's curvature — that is, the tendency of the trajectories in spacetime of freely falling bodies which start parallel to become non-parallel as the bodies move. The right-hand side expresses, in a covariant form, the distribution of energy density, momentum density, and their flow in spacetime.

We are not going to do a lot with this equation other than to examine several of its solutions. However, before getting into this, it is worth remarking on a couple of points.

- First, note that when working in Cartesian coordinates, the curvature tensor on the left-hand side has dimension $1/(\text{length})^2$. With that in mind, it is interesting to look at the numerical value of the constant[1] which connects the curvature tensor to the stress-energy tensor:

  $$\frac{8\pi G}{c^4} = 2.08 \times 10^{-43} \frac{\text{meter}^{-2}}{\text{J/meter}^3} \; . \tag{18.2}$$

  I've written the units to emphasize that this constant converts energy density (Joules per meter cubed) into curvature (inverse meters squared). Notice it takes a *lot* of energy density to produce a tiny amount of curvature. Osmium is the densest metal naturally found on Earth, at 22.6×10^3 kilograms per meter cubed (roughly three times the density of iron, and twice that of lead). Multiplying by c^2, we see that osmium has a rest energy density of 2.03×10^{21} Joules per meter cubed. But this density only produces 4.22×10^{-22} inverse meters squared of curvature. When you hear someone describe gravity as the weakest of the fundamental interactions, this is the essence of what they mean — we need a *lot* of energy density to curve spacetime. To get strong curvature, we need to go to regimes far beyond what we encounter on Earth.

  This doesn't mean that gravity is negligible though. Because "gravitational charge" — i.e., mass — only comes with one sign (there is no negative mass), its effects add up. Still, it's worth noting that every time you lift any object, electrochemical reactions

[1] In lecture on November 17, I wrote this with c^2 in the denominator rather than c^4. That was an error.

in a couple hundred grams of muscle tissue overcome the accumulated gravitational effects of 6×10^{24} kilograms of our planet.

- In discussions with some students after lecture, people seemed a little surprised by how *ad hoc* the derivation of the Einstein field equation seems to be. In essence, Einstein seems to have decided that the source should be $T^{\mu\nu}$, decided that the left-hand side should be a curvature tensor, then just matched $T^{\mu\nu}$ to a curvature tensor that is divergence free and has the right number of indices.

  This is not wrong! Einstein's original derivation of the Einstein field equation is indeed just as *ad hoc* as this makes it seem. Two remarks on this are in order:

  – First, it's worth bearing in mind that the ultimate arbiter of what description we should use for any physical interaction is *measurement*. You should therefore regard the Einstein field equation and its predictions as hypotheses to be tested. **Testing this hypothesis is still something being done today**, and in fact motivates quite a lot of modern research (including a bit of my own).

  – Around the time that Einstein formulated these field equations, other plausible formulations of relativistic gravity were also proposed. Those all were found to be flawed in important ways, failing experimental tests or turning out to have internal contradictions. General relativity can be regarded as the relativistic gravity theory that (so far, at least) best fits the data.

  – There's another way of deriving the field equation which is based on a variational principle, similar to the way that we apply variational principles to a Lagrangian in order to describe a body's motion. Though quite a bit beyond the scope of 8.033, it is worth remarking that this approach makes it clear that the Einstein field equation is, in a way that can be made precise, the *simplest* theory of relativistic gravity. A lot of research these days explores how general relativity may be, in a meaningful sense, itself an approximation to something deeper. This variational principle provides a foundation for exploring the nature of gravity.

There's a lot more we could say, but this will suffice for 8.033. The tack we are going to take from now on is to look at solutions of this equation and examine their consequences. I want to emphasize that *so far* we have not found any compelling evidence of shortcomings in general relativity's description of gravity, which is why this is often taught as "the" theory of relativistic gravity. But we keep looking.

18.2 Some example solutions and their significance

18.2.1 The "weak gravity" metric

Upon figuring out the field equation, Einstein developed its first solution. This is done by considering "weak" gravity — spacetime that is not *too* different from the metric of special relativity. This simplifies the curvature tensor, essentially by allowing us to approximate terms that are nonlinear in the spacetime metric as small enough that their influence can be neglected. The solution which emerges in this limit has 4 non-zero metric components:

$$g_{00} = -(1 + 2\Phi/c^2) \ , \quad g_{11} = g_{22} = g_{33} = (1 - 2\Phi/c^2) \ . \tag{18.3}$$

All other components of the spacetime metric are zero. The coordinates used here are

$$x^0 = ct\ , \quad x^1 = x \quad x^2 = y \quad x^3 = z\ . \tag{18.4}$$

The function Φ which appears in (18.3) is just the Newtonian gravitational potential. Outside a spherical body of mass M centered on the origin, it takes the form

$$\Phi = -\frac{GM}{r}\ , \qquad r = \sqrt{x^2 + y^2 + z^2}\ . \tag{18.5}$$

This metric works well when $\Phi \ll c^2$, which is a good description of spacetime almost everywhere in our solar system, for example.

18.2.2 The Schwarzschild metric

As mentioned at the end of the November 17 lecture, the first *exact* solution to the Einstein field equations was found by Karl Schwarzschild in 1916. It also has 4 non-zero metric components:

$$g_{00} = -\left(1 - \frac{2GM}{rc^2}\right)\ , \quad g_{11} = \left(1 - \frac{2GM}{rc^2}\right)^{-1}\ , \qquad g_{22} = r^2\ , \qquad g_{33} = r^2 \sin^2\theta\ . \tag{18.6}$$

All other components of the metric are zero. The coordinates used here are

$$x^0 = ct\ , \quad x^1 = r \quad x^2 = \theta \quad x^3 = \phi\ . \tag{18.7}$$

As we will discuss in an upcoming lecture, this describes *exactly* the spacetime outside of a spherically symmetric, non-rotating body of mass M. Schwarzschild found this solution essentially in his spare time while serving as an artillery officer on the eastern front during the First World War. Shortly after submitting this solution for publication, he died of an autoimmune disorder that most believe was sparked by an infection he contracted while serving in the trenches. The fact that this solution existed and was found so quickly shocked Einstein, who did not expect anyone would manage to find relatively simple exact solutions — certainly not so quickly after the field equations were developed.

This spacetime continues to play an important role in helping us to understand the limiting behavior of gravity; we will study it in some detail in coming lectures.

18.2.3 The Kerr metric

For decades, people wondered if there might be a more general exact solution than that provided by the Schwarzschild metric. What about near a body that is not spherical, or that is rotating? By the 1950s and 1960s, people were beginning to realize that one could take the Einstein field equation and treat it as a complicated differential equation that could be solved numerically, much as they were beginning to use computers to solve complicated differential equations describing things like fluid dynamics. As computers and computer programmers got more sophisticated, it became plausible model more interesting bodies. However, it seemed unlikely that a "closed form" solution for a body more complicated than spherical symmetry would ever be found.

That expectation held until 1963, when the mathematician Roy Kerr published the following glorious mess:

$$
\begin{aligned}
g_{00} &= -\frac{\Delta - a^2\sin^2\theta}{\Sigma} , \quad g_{11} = \frac{\Sigma}{\Delta} , \quad g_{22} = \Sigma , \\
g_{33} &= \left(\frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\Sigma}\right)\sin^2\theta , \\
g_{03} = g_{30} &= -\frac{2a\tilde{M}r}{\Sigma}\sin^2\theta ,
\end{aligned}
\tag{18.8}
$$

with all other metric components equal to zero, and where

$$
\Delta = r^2 - 2\tilde{M}r + a^2 \quad \Sigma = r^2 + a^2\cos^2\theta , \quad \tilde{M} = \frac{GM}{c^2} , \quad a = \frac{J}{Mc} . \tag{18.9}
$$

The coordinates used here are

$$
x^0 = ct , \quad x^1 = r \quad x^2 = \theta \quad x^3 = \phi . \tag{18.10}
$$

When Kerr originally published this solution, it wasn't actually clear what is meant. To be fair, he used a coordinate system which made it easier to solve the field equation, but made it less clear what the solution means; this form of the coordinates was published by Robert Boyer[2] and Richard Lindquist in 1967. If you set the parameter a to zero, it is not hard to show that the spacetime is identical to the Schwarzschild solution. After much study, it became clear that this solution describes a *black hole* with mass M and with spin angular momentum of magnitude $J = aMc$, oriented along the axis defined by $\theta = 0$. We will discuss this solution briefly, and perhaps explore quantitatively a simpler version of this spacetime[3], after discussing the Schwarzschild metric in more detail.

18.2.4 The Friedmann-Lemaître-Robertson-Walker metric

Finally, an exact solution that describes *all* of spacetime filled with a fluid of density ρ and pressure P is given by

$$
g_{00} = -1 , \quad g_{11} = a^2(t)/(1-kr^2) , \quad g_{22} = a^2(t)r^2 , \quad g_{33} = a^2(t)r^2\sin^2\theta . \tag{18.11}
$$

This again uses the coordinates

$$
x^0 = ct , \quad x^1 = r \quad x^2 = \theta \quad x^3 = \phi . \tag{18.12}
$$

The function $a(t)$ is the solution to the differential equations

$$
\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3c^2} - \frac{k}{a^2} , \quad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\rho + 3P) . \tag{18.13}
$$

(Overdot denotes d/dt.) The parameter k takes one of three values — $+1$, 0, or -1. Which value of k describes our universe is something that must be determined from data; unpacking this is kind of complicated.

[2] Boyer was tragically murdered, along with 17 other people, in an infamous mass shooting event at the University of Texas a few months before the paper's publication.

[3] If the spin angular momentum is "small" in a sense that can be made quantitative, the metric simplifies dramatically.

This solution was first found by the Soviet mathematician Alexander Friedmann in the early 1920s, although its significance was not broadly recognized prior to his death in 1925. Georges Lemaître, a Belgian priest and mathematician who earned a PhD in mathematics from MIT in 1923, rediscovered much of this solution in 1927. Via his efforts, people began to realize that this solution could be used as a tool for understanding the large-scale scale structure of the universe. Finally, Howard Robertson and Arthur Geoffrey Walker very thoroughly explored and described these spacetimes. Since the full cabal of discovers is a mouthful, this solution is often called the FRW (leaving out poor Lemaître) or FLRW metric.

The FLRW spacetime appears to give a good description of our universe on very long scales — tens of millions of light years, and over comparably long timescales. The "trick" is to come up with an appropriate description of the density and pressure that describes the "stuff" that characterizes the universe on such scales. This solution largely forms the foundation of the science of *cosmology*.

18.3 The Newtonian limit

18.3.1 The clocks of static observers

Let us begin our study of the consequences of general relativity with the solution that best describes spacetime near us: the "weak gravity" metric described in Sec. 18.2.1:

$$ds^2 = -\left(1 + \frac{2\Phi}{c^2}\right) c^2 dt^2 + \left(1 - \frac{2\Phi}{c^2}\right)\left(dx^2 + dy^2 + dz^2\right) \ . \tag{18.14}$$

We begin by thinking about the 4-velocity of an observer who is at rest in this spacetime; perhaps they are standing on the surface of the body that produces the gravitational potential Φ. How do we describe this observer's 4-velocity?

Since they are at rest in this spacetime, we require that $dx/d\tau = dy/d\tau = dz/d\tau = 0$. What remains is to figure out $dt/d\tau$. To deduce this, we insist that *exactly as in special relativity*, we must have $\vec{u} \cdot \vec{u} = -c^2$.

The reason we insist on this is because of Einstein's principle of equivalence: If we go into a freely falling frame, then everything behaves in spacetime *exactly as it did in special relativity*. We already know that $\vec{u} \cdot \vec{u} = -c^2$ in special relativity; and we know that the spacetime dot product is an invariant. We thus require that it have this form in **all** representations.

Enforcing this, we have

$$\begin{aligned}\vec{u} \cdot \vec{u} &= g_{\alpha\beta} u^\alpha u^\beta \\ &= -\left(1 + \frac{2\Phi}{c^2}\right) c^2 \left(\frac{dt}{d\tau}\right)^2 + 0 \\ &= -c^2 \ .\end{aligned} \tag{18.15}$$

Let's solve this for $dt/d\tau$, using the fact that the "weak gravity" metric requires $\Phi \ll c^2$:

$$\frac{dt}{d\tau} = \left(1 + \frac{2\Phi}{c^2}\right)^{-1/2} \simeq 1 - \frac{\Phi}{c^2} \ . \tag{18.16}$$

Let's take the source of the gravitational potential to be spherically symmetric and of mass M, so that $\Phi = -GM/r$. Let's consider two different observers: Observer 1 at height

r_1 (say, the surface of the Earth) has a clock which measures time τ_1; observer 2 at height $r_2 > r_1$ (some distance above the surface of the Earth) has a clock which measures time τ_2. Let's compare the rates at which their two clocks tick:

$$
\begin{aligned}
\frac{d\tau_2}{d\tau_1} &= \frac{dt/d\tau_1}{dt/d\tau 2} \\
&= \frac{(1 + GM/r_1c^2)}{(1 + GM/r_2c^2)} \\
&\simeq 1 + \frac{GM}{c^2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \ .
\end{aligned} \tag{18.17}
$$

Notice that since $r_2 > r_1$, this is positive: the clock of observer 2 ticks faster than the clock of observer 1. *This is exactly what we found based on our intuitive analysis of the light redshift effect.*

Before moving on, you might wonder — what does the coordinate t mean in this spacetime? We used it as an intermediate factor in order to compare the two observers' clocks, but the coordinate itself disappeared from the final analysis. To get some sense of this, notice that $dt/d\tau \to 1$ as $r \to \infty$. This means that the coordinate t is in fact proper time for an observer who is infinitely far away from the mass M. This tells us that t is time as measured on the clocks of very distant observers. We basically use t as a kind of "book-keeper" time; it's a time standard that everyone agrees on, no matter where they stand in spacetime. It facilitates making comparisons between different observers.

18.3.2 Falling down

Let's consider a body freely falling in the weak gravity spacetime (18.3. We begin by writing down the relativistic Lagrangian (per unit mass of the body) for this motion:

$$
L = \frac{1}{2}g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = -\frac{c^2}{2}\left(1 + \frac{2\Phi}{c^2}\right)\left(\dot{t}\right)^2 + \frac{1}{2}\left(1 - \frac{2\Phi}{c^2}\right)\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) \ . \tag{18.18}
$$

Here, an overdot denotes $d/d\tau$. Note that the potential Φ is independent of time, but depends on x, y, and z. Let's imagine a body that is falling along the z axis in this spacetime, so that $x = y = 0$, and see what applying the Euler-Lagrange equations tells us about the body's motion.

The equation of motion we need to examine is

$$
\frac{\partial L}{\partial z} - \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{z}}\right) = 0 \ . \tag{18.19}
$$

Let's evaluate these derivatives:

$$
\frac{\partial L}{\partial z} = -\left(\dot{t}\right)^2\frac{\partial \Phi}{\partial z} - \left(\frac{\dot{z}}{c}\right)^2\frac{\partial \Phi}{\partial z} \ , \tag{18.20}
$$

$$
\frac{\partial L}{\partial \dot{z}} = \left(1 - \frac{2\Phi}{c^2}\right)\dot{z} \ , \tag{18.21}
$$

$$
\begin{aligned}
\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{z}}\right) &= \left(1 - \frac{2\Phi}{c^2}\right)\ddot{z} - \frac{2\dot{z}}{c^2}\left(\frac{\partial \Phi}{\partial x}\dot{x} + \frac{\partial \Phi}{\partial y}\dot{y} + \frac{\partial \Phi}{\partial z}\dot{z}\right) \\
&= \left(1 - \frac{2\Phi}{c^2}\right)\ddot{z} - \frac{2\dot{z}^2}{c^2}\frac{\partial \Phi}{\partial z} \ .
\end{aligned} \tag{18.22}
$$

To get Eq. (18.22), we used the chain rule to expand the total derivative along the falling body's trajectory. We then used the fact that we are taking the body to fall only in the z direction to set $\dot{x} = \dot{y} = 0$.

The equation of motion we have derived appears to be a mess. Let's put all the pieces together and see what we get. For clarity, let's write all the overdot terms explicitly as $d/d\tau$:

$$-\left(\frac{dt}{d\tau}\right)^2 \frac{\partial\Phi}{\partial z} - \left(1 - \frac{2\Phi}{c^2}\right)\frac{d^2z}{d\tau^2} + \frac{(dz/d\tau)^2}{c^2}\frac{\partial\Phi}{\partial z} = 0\ . \tag{18.23}$$

Divide everything by $(dt/d\tau)^2$, and rearrange the terms:

$$\frac{d^2z}{dt^2} = -\frac{(1-(dz/dt)^2/c^2)}{(1-2\Phi/c^2)}\frac{\partial\Phi}{\partial z}\ . \tag{18.24}$$

Finally, using the fact that this metric requires $\Phi \ll c^2$, we can write this as

$$\frac{d^2z}{dt^2} = -\frac{\partial\Phi}{\partial z}\left(1 + \frac{2\Phi}{c^2} - \frac{(dz/dt)^2}{c^2} - 2\frac{\Phi(dz/dt)^2}{c^4}\right)\ . \tag{18.25}$$

The leading approximation to this equation is simply

$$\begin{aligned}\frac{d^2z}{dt^2} &= -\frac{\partial\Phi}{\partial z}\\ &= -\frac{GM}{r^3}z\ .\end{aligned} \tag{18.26}$$

This is nothing more than the Newtonian limit: the acceleration of a body falling in the spacetime (18.3) is given by minus of the gradient of the gravitational potential. Doing this calculation without assuming that the body is falling along the z axis yields the equation of motion,

$$\frac{d^2\mathbf{x}}{dt^2} = -\frac{GM}{r^3}\mathbf{x}\ . \tag{18.27}$$

This **exactly** reproduces Newtonian gravity.

It's worth noting that if this result had *not* been found, we would not be having this discussion today. Newtonian gravity works quite well over a wide range of important situations, and it was *necessary* for the relativistic version of gravity to reproduce Newton's successes. In fact, when we do a more complete derivation of the Einstein field equation, we use the fact that the theory should reproduce the Newtonian limit to pin down the constant of proportionality $8\pi G/c^4$ in the field equation.

What about those terms we've neglected in going from (18.25) to (18.26)? Notice that they introduce corrections to Newtonian gravity; notice also that each such term involves factors of $1/c^2$. That's a signal that they can be thought of as "relativistic corrections" to the leading result. For example, the first term we've neglected has a value at the Earth's surface of

$$\frac{2\Phi}{c^2} = \frac{2GM_{\rm Earth}}{c^2 R_{\rm Earth}} \simeq 1.38\times 10^{-9}\ . \tag{18.28}$$

This term introduces a roughly part per billion correction to gravitational acceleration. The second term is of order the small body's speed squared divided by c^2; the third is the product of those two corrections.

For the vast majority of applications, those corrections are negligible — indeed, measuring them at all is not easy. However, Einstein thought it might be interesting to include their influence and see what effect they have on the motion of bodies moving under the influence of gravity. He was motivated by the fact that for centuries people had been wondering how to resolve a mystery regarding Mercury's orbit. It was well known that an orbit in Newtonian gravity — i.e., an orbit governed by Eq. (18.27) — would be a closed ellipse, *if* we had a single small body orbiting a single large body. It was also well known that if the system was more complicated than this simple two-body setup, then the ellipse wouldn't quite close — it would precess, with the axis along its long direction slowly rotating with time.

Mercury's orbit is determined mostly by the gravity of the Sun, but it is perturbed by other planets in the solar system — especially Venus and Earth (which are fairly close by), and Jupiter (which is very massive). During the 19th century, a lot of mathematical techniques were perfected figuring out to account for the actions of these planets on Mercury's orbit. After a lot of back and forth, the consensus emerged: Mercury's orbit should precess by 5556 arcseconds per century.

Sadly for the natural philosophers of the 19th century, the data did not quite bear this out. Over many decades of observation it became clear that Mercury's orbit precessed a little too fast — the data give us a rate of 5599 arcseconds per century. A discrepancy of 43 arcseconds per century was clearly present in Mercury's orbit data.

Many hypotheses were advanced to explain this, including the idea that a planet provisionally named Vulcan[4] occupied an orbit very close to the Sun, inside Mercury's orbit. None of them worked. Einstein was curious what happens if he turned the crank on Eq. (18.25), including terms which are of order $1/c^2$. With some effort, and focusing on a bound orbit in the spacetime (18.3), one can show that the equation of motion becomes

$$\frac{d^2\mathbf{x}}{dt^2} = -\frac{GM}{r^3}\left(1 + \frac{|\mathbf{v}|^2}{c^2}\right)\mathbf{x} + \frac{4GM(\mathbf{x}\cdot\mathbf{v})\mathbf{v}}{c^2 r^3} + \mathcal{O}\left(\frac{1}{c^4}\right) . \tag{18.29}$$

(The $\mathcal{O}(1/c^4)$ in this equation means that the next term, which we are ignoring, involves things that scale with $1/c^4$.) With a little effort, one can show that an orbit governed by this equation of motion is described by a precessing ellipse. When applied to Mercury's orbit, the rate at which the angle of the orbit's ellipse rotates is given by

$$\frac{d\phi}{dt} = \frac{6\pi G M_\odot}{a(1-e^2)Pc^2} . \tag{18.30}$$

In this equation, $M_\odot = 1.99 \times 10^{30}$ kg is the mass of the sun; $a = 57.9 \times 10^6$ km is the semi-major axis of Mercury's orbit; $e = 0.2$ is the eccentricity of the orbit; and $P = 88$ days is the period of the orbit. Plugging all these numbers in, using 36,524 days per century, we find the rate of advance of Mercury's orbital ellipse due to relativistic corrections:

$$\frac{d\phi}{dt} = 0.000208\,\text{radians/century} . \tag{18.31}$$

There are 2π radians in 360 degrees; there 3600 arcseconds in one degree. Hence, there are $360 \cdot 3600/2\pi = 206{,}265$ arcseconds per radian. Converting units, Einstein found that general relativity's prediction for the "extra" precession of Mercury's orbit is

[4] Proposed *way* earlier than Gene Roddenberry's time.

$$
\begin{aligned}
\frac{d\phi}{dt} &= (0.000208\,\text{radians/century})\left(2.063\times 10^5\,\text{arcseconds/radian}\right) \\
&= 42.9\,\text{arcseconds/century}\ . \qquad (18.32)
\end{aligned}
$$

Further refinements to these numbers only improves the fit. In one fell swoop, Einstein managed to explain a phenomenon that puzzled many of the most significant mathematicians and physicists of history. No wonder that in a letter to his friend and colleague Paul Ehrenfest shortly after completing this calculation, he wrote "I was beside myself with ecstasy for days."

18.4 Addendum: Other attempts to make relativistic gravity

As emphasized at the beginning of this discussion, we should take general relativity as described by the field equation $G^{\mu\nu} = (8\pi G/c^4)T^{\mu\nu}$ as a *hypothesis*, one that must be tested by comparing with data. It was not inevitable that we would end up with what we now call general relativity. Here is a brief discussion of a few alternates that were considered, and why we they didn't hold up.

- Motivated by the idea that one can needs to make $\nabla^2\Phi = 4\pi G\rho_M$ something that makes sense in Lorentz frames, the Swedish/Finnish physicist Gunnar Nordström proposed that gravity acts via a scalar field Φ which, in the language we are now using, satisfies the differential equation
$$\Box\Phi = -\frac{4\pi G}{c^4}\Phi^5\eta_{\alpha\beta}T^{\alpha\beta}\ . \qquad (18.33)$$
(Note, it's possible I have botched a few factors! In particular, I haven't carefully checked the powers of Φ on the right-hand side. The form in which this theory appears in textbooks involves using some quantities which would be a big detour for us to introduce and discuss here; I don't guarantee that I've translated this with 100% accuracy.) With a little effort, it can be shown that this yields an equation of motion that looks like
$$\frac{d(\Phi u_\alpha)}{d\tau} = -\frac{\partial\Phi}{\partial x^\alpha}\ . \qquad (18.34)$$
In the limit of $\Phi \ll 1$, this reproduces Newtonian gravity, and correctly produces the redshifting of light. However, it turns out to get Mercury's precession wrong; and, it predicts that light rays do not change direction under the influence of gravity. The bending of light by gravity was a particularly important early triumph of Einstein's version of relativistic gravity.

- Motivated by the idea that $\mathbf{F}_g = -Gm_1m_2\mathbf{x}/r^3$ looks an awful lot like the Coulomb interaction, perhaps we can define a quantity like the Faraday tensor which describes gravity. In short, one might wish to construct a Maxwell-equation-like theory of gravity.

  This can be done, but the result turns out to be *theoretically* inconsistent. Whenever one makes an interaction relativistic, one finds that it predicts the interaction produces radiation. This is a simple consequence of causality: If we "shake" the source of the interaction (e.g., charges for electric and magnetic fields, masses for gravity), the outcome of this shaking can be communicated to distant observers no faster than the speed of light. Indeed, all relativistic theories of gravity predict that some form of gravitational radiation must exist.

> When we do this for a “Maxwell-like gravity,” the radiation that it produces has a very weird feature: the radiation that it produces has *negative energy density*. This means that in this theory, if I have a dynamical system that produces radiation, it carries away “negative energy” from the system. Taking away “negative energy” is the same thing as *adding* energy. The dynamics that made the system radiate in the first place thus become **more** energetic — making the radiation have higher amplitude, which means they carry away **more** negative energy, thus making the system even **MORE** energetic.
>
> Such a description of gravity turns out to be catastrophically unstable — any dynamics would almost immediately become grow without bound, destroying the system. Since we do not observe this (indeed, since we exist in order to observe that this does not happen), we reject the Maxwell-like theory of gravity. (Details of this analysis can be found in exercise 7.2 in the textbook *Gravitation* by Misner, Thorne, and Wheeler. It is not a simple exercise!)

Though ideas of this kind didn’t hold up, we haven’t stopped thinking about ways in which Einstein’s general relativity may not quite meet the mark. Precisely because gravity is the weakest fundamental interaction, it is extremely difficult to test. It’s worth noting that the gravitational constant G is the least precisely determined of the main “fundamental constants” of nature — although the product of G with certain masses is quite well known, simply because that product is what enters many observable formulas. For example, although G is known to about 5 digits, $GM_\odot$ is known to about 10 digits.

Thinking about plausible modifications to general relativity, and coming up with experimental methods for testing them, is among the topics that are at the vanguard of modern physics research.