Massachusetts Institute of Technology
Department of Physics
8.033 Fall 2021

Lecture 21
Data on strong gravity

21.1 Overview; Kerr versus Schwarzschild

Having discussed in some detail the features which make strong-gravity spacetimes special, we turn now to what data and observations have taught us.

The most interesting feature to observe would of course be an event horizon. However, the horizon is hard to observe since, by definition, its existence is the signpost of an absence — a "one-way membrane" from which we can get no information. One can imagine looking for spacetimes that describe a very dense, massive body, but that appears to lack a well-defined surface. That indeed has been done, and is responsible for providing many of the black hole *candidates* studied by astronomers over the years. In this set of lecture notes, we will focus our discussion of measurements that have provided evidence for other features associated with motion in very strong gravity that we have discussed:

- *Non-Keplerian orbits*: As noted in Lecture 20, strong-field orbits are not closed ellipses in general, but show rather more complicated patterns of motion. This fundamentally arises from the fact that, in the time takes an orbiting body to move from minimum radius to maximum radius and back, the body moves through more than 2π radians. At its core, this is the same effect that leads to the 43 arcseconds per century of Mercury's anomalous perihelion precession; in very strong-field spacetimes, the effect is turned up quite a bit louder.

- *Unstable orbits*: No stable circular orbits exist for $r \le 6GM/c^2$.

- *The light ring*: The gravitational bending of light becomes so severe that a light ray can in principle loop around forever (though in practice, because this is an unstable orbit, we expect it to loop around a few times at most before zooming out).

We did this analysis and found all these effects in the Schwarzschild spacetime. However, the generic solution that we expect in Nature is the Kerr solution. This introduces some interesting complications:

- *Frame dragging*: As you showed on problem set #9, near a rotating black hole, spacetime "wants" to pull you along, so that you move in the same sense in which the black hole is rotating. When you get close enough, it becomes impossible to resist this motion, and you are forced to move in the same rotational sense as the black hole, no matter how strongly you oppose it. This amplifies the non-Keplerian features that we saw in the case of a Schwarzschild metric.

- *The properties of unstable orbits depend on orbit orientation*: There are unstable orbits in the Kerr spacetime, analogous to the orbit at $r = 6GM/c^2$ that we found for

Schwarzschild. However, when the black hole is spinning, the radius of these orbits varies depending upon the orientation of the orbit with respect to the spin axis. Figure 1 (the curves labeled "material body orbit") shows what we find for orbits that are in the black hole's equatorial plane (i.e., orbits that have $\theta = \pi/2$). Orbits which are *prograde* move in the same sense as the black hole's rotation; those which are *retrograde* move in the opposite sense of the rotation. Notice that as the black hole's spin increases from $a = 0$ (which is the same thing as Schwarzschild) to $a = GM/c^2$, the radii of these two possibilities diverges quite a bit.

- *Light rings depend on orbit orientation*: Just as the unstable orbit's position varies with orientation, so does the radius of the light ring. Figure 1 also shows the radii of the light ring associated with orbits that have $\theta = \pi/2$, and it also splits into a prograde and a retrograde branch.

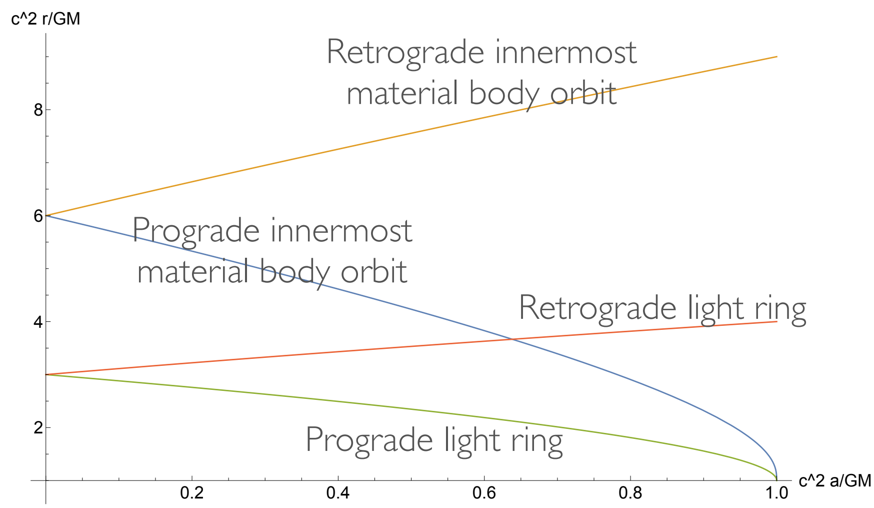

Figure 1: Important orbital radii in the Kerr spacetime as a function of black hole spin parameter a. The two "material body orbit" curves show the radii at which circular orbits become unstable. The "prograde" curves traces out the radius of orbits which move in the same sense as the black hole's spin; the "retrograde" curves traces out this radius for orbits which move opposite to the black hole's spin. The two light ring curves do the same things for the radius at which light rays are captured onto orbits.

Conceptually, it's not very difficult to generalize everything we did for Schwarzschild to Kerr. Getting the details right, however, are somewhat involved. We won't go through the details here, but want to emphasize that these additional complications should be borne in mind as we examine the data we have on strong-gravity systems.

21.2 Gravitational radiation

Light is often hard to observe from very strong-gravity systems. The most interesting part of the system is dark; any light comes from objects or matter that are moving near or orbiting around the darkest bit. Such sources of light are often "buried" in dense astronomical environments with lots of other bodies and matter around, which makes it hard for their light to get out. Very bright, intense, high-energy light may be generated deep in these spacetimes, but the light can be highly scattered or absorbed by other matter, making it difficult for us to observe it and to use it to study these spacetimes.

"Light" is being used here as shorthand for electromagnetic radiation — oscillating disturbances to electric and magnetic fields which propagate across spacetime. In the past several years, decades of effort have come to fruition to use another form of radiation: *gravitational* radiation, or gravitational waves (which we'll abbreviate GWs). GWs are another consequence of the theory of relativity. Their existence follows from the fact that any relativistic theory involving fields which act at a distance *must* predict that the field itself can radiate. This radiation reflects how changes to the field propagate across spacetime as the field's source moves around.

If spacetime is nearly that of special relativity, we can write $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$. Run this through the Einstein field equations $G^{\mu\nu} = (8\pi G/c^4)T^{\mu\nu}$, discard all terms that are of order h^2, and the result is[1]

$$\Box h_{\alpha\beta} = -\frac{16\pi G}{c^4} T_{\alpha\beta} \; . \tag{21.1}$$

The "weak gravity" metric that we discussed a few lectures ago is a solution of this equation when the time variations are zero (so that $\partial(\text{anything})/\partial t = 0$). When the source *is* time varying, the solutions to this equation are time-varying metric components $h_{\alpha\beta}$ that propagate across spacetime. An example of an allowed solution is one which takes the form

$$h_{\alpha\beta} \doteq \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h(z-ct) & 0 & 0 \\ 0 & 0 & -h(z-ct) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{21.2}$$

The function $h(z - ct)$ which appears in these tensor components is related to the dynamics of the gravitating source; we'll talk about what it looks like in a moment. The key thing to emphasize for us right now is that the influence of this function on spacetime can be measured by looking at light propagating in the x and y directions.

Consider light which propagates in the x direction. How much time does it take to travel a distance L? We calculate this using the metric. Use the fact that light has a 4-momentum which obeys $\vec{p} \cdot \vec{p} = 0$, and that it propagates in the x direction:

$$0 = \vec{p} \cdot \vec{p} = (\eta_{tt} + h_{tt}) \left(c \frac{dt}{d\lambda} \right)^2 + (\eta_{xx} + h_{xx}) \left(\frac{dx}{d\lambda} \right)^2 \; . \tag{21.3}$$

Rearranging this, we solve for dt/dx as the light propagates:

$$\begin{aligned} \frac{dt}{dx} &= \frac{1}{c}\sqrt{1 + h(z-ct)} \\ &\simeq \frac{1}{c}\left[1 + \frac{1}{2}h(z-ct)\right] \end{aligned} \tag{21.4}$$

[1] Actually, I am eliding an important subtlety involving what is called the "choice of gauge." Just as one can adjust the scalar potential ϕ and the vector potential $\mathbf{A}$ of electrodynamics in such a way as to leave the fields $\mathbf{E}$ and $\mathbf{B}$ unchanged, so one can adjust the metric-like quantity $h_{\alpha\beta}$ but leave its associated *curvature* tensors unchanged. For the purposes of 8.033, this subtlety is a tangent that we can skip over.

Here, we've assumed that the function $h \ll 1$; as we'll see shortly, this is a reasonable assumption. We now integrate up to compute the time it takes for light to propagate this distance:

$$\begin{aligned}\Delta t_x &= \frac{1}{c}\int_0^L \left[1 + \frac{1}{2}h(z - ct)\right] dx \\ &\simeq \frac{L}{c}\left[1 + \frac{1}{2}h(z - ct)\right] . \end{aligned} \tag{21.5}$$

On the last line, we imagine that in the time it takes light to travel a distance L, the function $h(t - z)$ changes by very little. If this is not correct, then some important details of the analysis change, but the final result is quite similar.

Imagine that while light travels in the x direction, light also travels a distance L in the y direction. Repeating this calculation along the y axis, we find

$$0 = \vec{p} \cdot \vec{p} = (\eta_{tt} + h_{tt})\left(c\frac{dt}{d\lambda}\right)^2 + (\eta_{yy} + h_{yy})\left(\frac{dy}{d\lambda}\right)^2 , \tag{21.6}$$

or

$$\frac{dt}{dy} = \frac{1}{c}\sqrt{1 - h(z - ct)} \tag{21.7}$$

$$\simeq \frac{1}{c}\left[1 - \frac{1}{2}h(z - ct)\right] . \tag{21.8}$$

Integrating up, this yields

$$\Delta t_y = \frac{1}{c}\int_0^L \left[1 - \frac{1}{2}h(z - ct)\right] dy \tag{21.9}$$

$$\simeq \frac{L}{c}\left[1 - \frac{1}{2}h(z - ct)\right] . \tag{21.10}$$

So in the x direction, the light travel time is a bit longer than it would be without the gravitational wave; in the y direction, it is a bit shorter. For the most interesting sources, the function h is sinusoidal, so it oscillates — but it always does so in such a way that the light travel time is long in one direction, short in the other.

The function h itself depends on the nature of the source. Solving the "linearized" Einstein field equation (21.1), we find that the leading-order solution describing radiation looks like[2] this for a source that is a distance r away from us:

$$h_{00} = 0 , \quad h_{0j} = 0 , h_{jk} = \frac{2}{r}\frac{G}{c^4}\frac{d^2\mathcal{I}_{jk}}{dt^2} . \tag{21.11}$$

This is called the *quadrupole formula* for gravitational waves, because the tensor $\mathcal{I}_{jk}$ is related to the quadrupole moment of mass and energy distributed in the source:

$$\mathcal{I}_{jk} = \int \rho_M(\mathbf{x})\left(x_j x_k - \frac{1}{3}\delta_{jk}|\mathbf{x}|^2\right) d^3x . \tag{21.12}$$

[2]Following the previous footnote about eliding some details having to do with gauge, those details have an influence on the precise details of the following equation too. Nonetheless, this gets all the details right up to some overall factors that reflect how the waves "look" from different viewing angles.

(Here $\rho_M \equiv \rho/c^2$ is the mass density distribution.) Those of you who have studied some advanced electrodynamics may be reminded of the *dipole* formula for electromagnetic radiation, which shows us how the radiative potential that arises from a dynamical charge source varies as a single time derivative of a source's charge dipole momentum.

To get an idea of the size of the effect that we expect for gravitational waves, let's make a rough estimate for how big a typical component of the wave tensor will be. The typical magnitude of a non-zero component of $\mathcal{I}_{jk}$ is $\sim MR^2$, where M is the amount of mass that is dynamical in the system, and R is the amplitude of its motion. Take two time derivatives, and assume that the mass is bound into some kind of orbital motion; you find that the typical magnitude of $d^2\mathcal{I}_{jk}/dt^2$ is $\sim Mv^2$, where v is the speed associated with that bound orbital motion. Combine this with Eq. (21.11) and we get the typical magnitude we might expect for a GW:

$$h_{jk} \sim \frac{GM}{c^2 r}\frac{v^2}{c^2} . \tag{21.13}$$

Let's imagine a source that involves 50 solar masses in orbital motion with speeds typically near 10% of the speed of light; imagine that source is located about a billion light years away. Using these numbers, we find

$$h_{jk} \sim 10^{-22}\left(\frac{M}{50\,M_\odot}\right)\left(\frac{10^9\,\mathrm{lyear}}{r}\right)\left(\frac{v}{0.1\,c}\right)^2 . \tag{21.14}$$

(The symbol $M_\odot$ stands for 1 solar mass.) This sets the stage for the magnitude of the timing effect we need to be able to detect — roughly a part in 10^{22} or so.

How one can actually make such measurements is a topic for a whole other course. Suffice it to say that it can be done; it has been done; and it is not easy. Let us move on to discussing what we learn when we can measure these waves.

21.3 Observing objects in orbit about black holes

Let us turn now back to what we can (and do!) observe. The most important data comes from observing objects that orbit very massive things. Some of the most compelling examples have been observations of stars which orbit some kind of a very massive but dark object. Over the course of about 30 years, astronomical techniques have made it possible to resolve stars moving in the very innermost regions of the galactic center. What these objects have showed us is that roughly half a dozen stars move on orbits very close to a big "something," with with orbital properties that noticeably change over the course of several years.

Several of these stars complete their orbits in ten or so years, making it possible to use them to precisely measure the mass of the object that they orbit. The mass we find turns out to be

$$M \approx 4 \times 10^6\,M_\odot . \tag{21.15}$$

So these stars are orbiting around 4 million solar masses of *something*. However, there is *no* object visible that these stars orbit around — whatever that 4 million solar mass "thing" might be, it is dark and it is massive. At least one of those stars is now seen to undergo orbit precession in a way that aligns perfectly with the "non-Keplerian" aspect of black hole orbits that we discussed in Lecture 20; rather than advancing by 43 arcseconds per century like Mercury, its orbital ellipse advances by about 10° per orbit.

This object at the center of our galaxy has long been perhaps the most striking example of a spacetime that describes something that is really massive but dark that we have studied with telescopes, though there are quite a few others. In the past several years, some of the most compelling data probing such spacetimes has come from gravitational-wave observations. Suppose two objects are in circular orbit around one another. The gravitational waves that they generate carry away energy and angular momentum from the system. This causes the objects to fall closer toward one another. When this happens, they move faster, generating stronger gravitational waves, causing them to fall toward one another even faster. The result is a characteristic *chirping* waveform. This "chirping" continues until the two bodies come so close to one another that there no longer exists a stable circular orbit. When this happens, the two objects plunge together. Figure 2 shows an example of what a waveform in this scenario looks like.

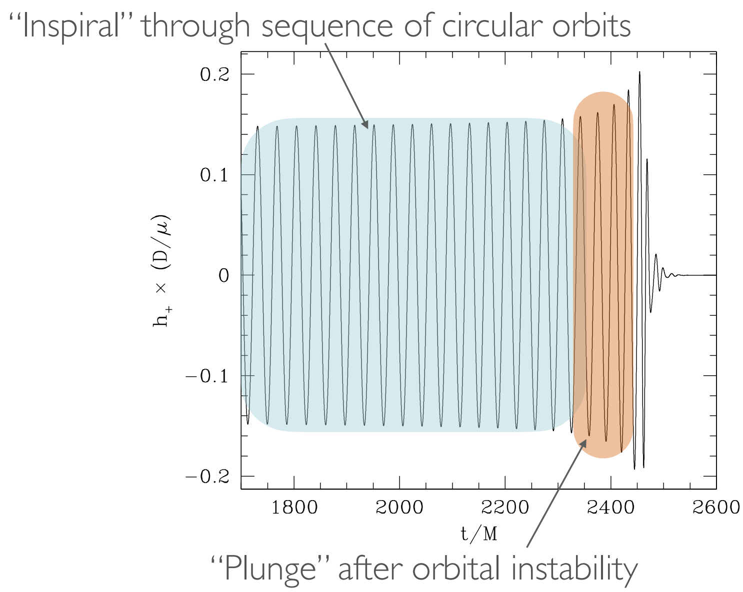

Figure 2: An example of a theoretical model of a gravitational waveform computed for bodies in circular orbit about one another. The region highlighted in light blue shows the waveform over the range of orbits for which the system is slowly evolving through a sequence of circular orbits; the area in orange corresponds roughly to the waves after the members have come so close to one another that stable circular orbits no longer exist. The last few decaying cycles are described in more detail below. Note the units are a little different from what we generally use in this class; multiply M by G/c^3 on the horizontal axis, and multiply μ [the system's reduced mass, $\mu \equiv m_1 m_2/(m_1 + m_2)$] by G/c^2 on the vertical. D is the distance to the binary from the detector that observes this waveform.

This figure shows us one example of the gravitational waveform produced by two bodies orbiting each other; this example was computed for a system with a mass ratio of 10. In this case, we see a train of cycles in which the amplitude starts out changing fairly slowly. This is what we expect at this mass ratio when the system is evolving through a sequence of stable circular orbits. The amplitude starts changing much more rapidly when the members of the binary become close enough that a stable orbit no longer exists, around $t \sim 2300GM/c^3$. At this point, they plunge toward one another, accelerating very rapidly, generating very strong waves at least until they merge into one object. (The nature of the final damped cycles at the end we describe a bit further below.)

We have been able to compute waveforms like that shown in Fig. 2 for quite a while, but measuring these waves is a challenge, thanks to the fact that the effect we are trying to measure amounts to a timing variation of about 1 part in 10^{22}. Hard work, much of it done by colleagues here at MIT, steadily improved the sensitivity of the antennae which can measure this effect. For many of us, the world changed in Fall of 2015, when the two detectors of the LIGO Laboratory recorded the signals shown in the top panels of Fig. 3.

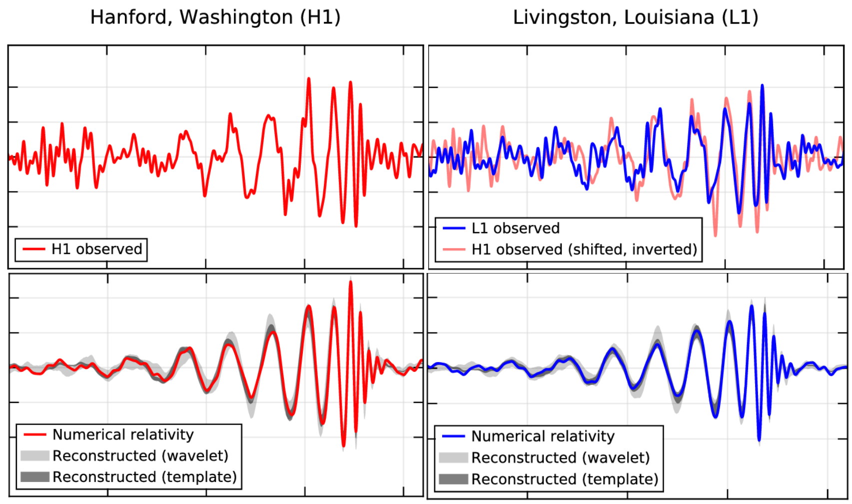

Figure 3: The first directly observed gravitational wave event. The data in the top two panels (which show the traces of h picked up by the two LIGO detectors) match superbly to theoretical models of the waves produced by black holes of mass $29\,M_\odot$ and $36\,M_\odot$ merging to produce a single black hole of $62\,M_\odot$.

These data show the gravitational waveform that was picked up by the two antennae run by the LIGO laboratory (one in eastern Washington, one in a pine forest in Louisiana). It also shows the waveforms that are predicted by solving the Einstein field equations. (Note that these waveforms are the output of *much* more complicated calculations than we have explored in 8.033! Because there are *two* massive bodies, the spacetime is more complicated than the Schwarzschild or Kerr spacetimes we have been studying. Supercomputer simulations are needed to model these solutions in general, though a lot of insight comes from careful "analytic" modeling as well.) The agreement between theory and data is superb. The conclusion is that a black hole of mass $29\,M_\odot$ merged with a black hole of $36\,M_\odot$, leaving a $62\,M_\odot$ remnant black hole[3] behind.

Since that first discovery, the two LIGO instruments plus the Virgo antenna in Pisa Italy (which was commissioned and joined observations a year or so later) have measured nearly 80 such merging black hole pairs, as well as a few events that involve neutron stars as well. Our universe appears to be full of black holes.

[3] You might notice that some mass appears to be missing — $36 + 29 \neq 62$. In fact, an amount of energy equal to $3\,M_\odot c^2$ was lost due to gravitational radiation produced by the system. Most of that energy was lost in roughly 0.1 seconds. If that energy had been radiated in light rather than GWs, then during that second, this system would have shined more brightly than several hundred *billion* Milky Way galaxies.

21.4 Observing the light ring

The data we have briefly discussed in this lecture covered a few of the features of strong-gravity orbits that we discussed previously — the non-Newtonian orbit shapes that are seen in the motion of stars in our galactic center, and the orbital instability. It should be emphasized that as detectors get more sensitive, and new instruments make it possible to observe different bands[4] of gravitational waves, we expect to be able to "watch" systems evolve through a wide range of orbits. Orbits with substantial eccentricity are ones that are likely to be especially interesting, and to carry a lot of information that will allow us to probe the nature of these systems.

But what about that light ring? The light ring is one of the most striking predictions of motion in black hole spacetimes. Perhaps the biggest challenge here is one of scale. Consider the black hole in the center of our galaxy, with a mass $M \approx 4 \times 10^6\, M_\odot$. How big do we expect the ring to be in this case?

Recall that for light moving the Schwarzschild spacetime, we expect the ring to be of radius $b = 3\sqrt{3}GM/c^2$. For the black hole in the center of galaxy, this translates to $b \approx 30$ million kilometers. That sounds big! — but the ring is in the center of our galaxy, which is about 27,000 light years from our solar system. Such a ring would have an angular diameter on the sky of

$$\delta\theta_{\rm ring} = \frac{2 \times 30,000,000\,\mathrm{km}}{27,000\,\mathrm{lyear}} \simeq 2.4 \times 10^{-10}\,\mathrm{radians} \approx 0.05\,\mathrm{milliarcseconds}\ . \tag{21.16}$$

This is an *extraordinarily* small angle; recall there are 3600 arcseconds in a degree, and this is smaller than an arcsecond by a factor of 20,000. Further complicating this is that we need to see "through" a lot of intervening gas and plasma, which tends to scatter electromagnetic radiation. By carefully studying the properties of all that "stuff" which is in the way, a team of astronomers deduced that radiation with a wavelength of about 1 millimeter was the best choice to look at the core of our galaxy, as well as the cores of a few nearby galaxies. The galaxy M87 was of particular interest — it is 1000 times farther away than the center of our galaxy, but appears to host a black hole that is about 1000 times more massive. The factors of 1000 cancel out as far the angular size is concerned, and the light ring is similar in size to what we estimated above.

If you're trying to resolve something with an angular size $\delta\theta$, Rayleigh's criterion teaches us that the diameter D of the telescope we need to use is related to the wavelength λ of the radiation we are measuring according to

$$\delta\theta = \frac{1.22\lambda}{D}\ . \tag{21.17}$$

Plugging in $\lambda = 10^{-3}$ meters, and using the $\delta\theta_{\rm ring}$ we estimated above, we find that we need $D = 5000$ kilometers — comparable to the radius of the Earth!

This may seem challenging — and it is. However, we don't need a single telescope of this size; we "just" need to have an array of telescopes that are *separated* by this distance. If we

[4]Currently active instruments are sensitive to gravitational waves which oscillate in the frequency band several $\times\ 10\,\mathrm{Hz} \lesssim f \lesssim 1000\,\mathrm{Hz}$. Sources which radiate in this band tend to have masses similar to stars — solar masses up to about a hundred or so solar masses. Planned detectors will broaden this; your lecturer is particularly excited about space-based instruments which will be sensitive in a band of about $10^{-4}\,\mathrm{Hz} \lesssim f \lesssim 0.1\,\mathrm{Hz}$. Waves in this band will come from sources of millions of solar masses, like the kind of black holes that appear to exist in the cores of many galaxies, including our own.

can then combine the data from all these telescopes in just the right way (and doing this requires that we know when each bit of data arrived with a precision better than $\delta t < \lambda/c$, and we need to know the distance between telescopes with a precision of about λ), then we can treat all the data as coming from a single telescope whose size is given by the different telescopes' separations.

Such measurements were done by a multi-month observing campaign focusing on the black hole candidate at the center of M87 by a collaboration called the "Event Horizon Telescope." They first announced their results in Spring 2019. For our purposes, perhaps the most exciting result is the one shown in Fig. 4.

Figure 4: Emission in the inner few dozen microarcseconds at the center of the galaxy M87.

This is what a light ring actually looks like for emission deep inside a strong-gravity spacetime. Note that it's thicker than the (rather idealized) picture that we sketched in a previous lecture. This is in part because the illumination which provides the light we observe is itself kind of "lumpy," brighter in some areas than others, and appears to be orbiting around the black hole. Also, the telescope's resolution blurs things out somewhat. Bearing those corrections to our ideal picture in mind, this ends up having exactly the characteristics expected for a black hole light ring in general relativity.

In addition to this light ring, the light ring has an influence on the gravitational waves that we have been measuring since 2015. Look again at the final few cycles of the waveforms shown in Figs. 2 and 3. Notice that they very rapidly decay away, oscillating several times as they do so. These final few cycles oscillate at a period that very closely corresponds to what we expect for light orbiting in the light ring. Gravitational waves propagate across spacetime just as light does, and so **gravitational waves can be trapped in the light ring** just as light rays can get trapped.

That in fact is what we are seeing in those final gravitational wave cycles. Those final cycles can be thought of as gravitational waves from the coalescence that orbit around a few times in the spacetime of the remnant black hole that is left over at the end of coalescence. Because that light ring is an unstable orbit, those last gasps of radiation leak away, gradually reducing in amplitude as more and more of that trapped radiation leaves the strong-field region of the spacetime.

Both light and gravitational-wave observations have confirmed all those crazy features associated with strong-gravity spacetimes. All the evidence to date tells us that black holes exist, and that they have all the properties that general relativity tells us they should have. Albert Einstein's strong-gravity spacetimes hold up to our observations.