Massachusetts Institute of Technology
Department of Physics
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Lecture 22
Our universe at large

22.1 Does $T^{\mu\nu} = 0$ describe our universe?

Strong-gravity spacetimes tell us about "compact" bodies, things that can be localized to some spatial region. They reproduce Newtonian gravity, and they introduce new behavior that (so far, at least!) all fits the data. However, these spacetimes are "asymptotically flat": when we go very far away from the source of mass in the spacetime, we find $ds^2 \to -c^2dt^2 + dx^2 + dy^2 + dz^2$. Does this behavior describe our universe? Spacetimes for which this true all solve the Einstein field equations *if* $T^{\mu\nu} = 0$. Is this an accurate description of our universe?

The answer to this, very clearly, is **no!** Looking out, we see our galaxy, other galaxies, clusters of galaxies, light, gas. Indeed, on the very largest scales, the universe appears to be a uniform fog of matter and radiation, limiting to a haze of microwaves known as the "cosmic microwave background," or CMB, at the largest distances that we are able to probe. However, an interesting property of what we see is that the universe is quite uniform on the largest scales. For example, on the very largest scales we can measure, variations in the CMB are a fraction of about 10^{-5} of its mean level[1]. Things become clumpier on smaller scales because gravity tends to make things clump up.

On the very largest scales — larger than about $10 - 100$ Megaparsecs[2] — we can think of our universe as a perfect fluid. This may seem crazy, but it is an acceptable treament as long as we focus on scales where matter's granularity has no effect. It's kind of the way we treat water as a fluid, even though we know it is made of individual molecules. On large enough scales, the granularity of water cannot be perceived; on large enough scales, the granularity of stars and galaxies cannot be perceived.

22.2 A spacetime for the large-scale structure of the universe

Although the universe is uniform in all spatial directions on the largest lengthscales, it is *not* uniform in time. Light travels at finite speed, so large distances are seen at earlier times. What we see at earlier times is a universe that was much denser than today.

To describe the large-scale structure of our universe's spacetime, we want to use a metric that is uniform in space, but not in time. It can be proven that the most spatially symmetric spacetime has the form

$$ds^2 = -c^2dt^2 + a^2(t)\left[\frac{d\bar{r}^2}{1 - k\bar{r}^2} + \bar{r}^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right] . \tag{22.1}$$

[1] After correcting for a Doppler effect. The CMB defines a preferred rest frame, and we are moving with respect to that rest frame due to the motion of our solar system with out galaxy, plus the infall of our galaxy toward the local cluster of galaxies.

[2] 1 parsec = 3.26 lightyears. This is a unit of distance that is particularly useful in astronomy, because it arises directly from measurements we can make using parallax.

This is a "Robertson-Walker" spacetime. It has the following important properties:

- The function $a(t)$ is the *scale factor*, and controls the physical scale associated with distance between two objects. If $k = 0$, the distance between $(\bar{r}_1, \theta, \phi)$ and $(\bar{r}_2, \theta, \phi)$ is

$$L = a(t)\left[\bar{r}_2 - \bar{r}_1\right] \ . \tag{22.2}$$

  Notice that if two objects are at spatial rest in the coordinate system (so that $\bar{r}$, θ, and ϕ are all constant) then the physical distance between them is nonetheless changing if $a(t)$ changes with time.

- The coordinate $\bar{r}$ is a dimensionless radial coordinate. For $k = 0$, $a(t)\bar{r}$ is essentially just our "normal" spherical distance.

- The parameter k is called the "spatial curvature" parameter, and takes the value -1, 0, or 1. For $k = 1$, we define

$$\frac{d\bar{r}}{\sqrt{1-\bar{r}^2}} = d\chi \mapsto \bar{r} = \sin\chi \ . \tag{22.3}$$

  In this case, the value of $\bar{r}$ is bounded: we can never exceed $\bar{r} = 1$. This describes a *closed universe*: the physical separation between objects has a maximum at each moment in time.

  For $k = -1$, we define

$$\frac{d\bar{r}}{\sqrt{1+\bar{r}^2}} = d\chi \mapsto \bar{r} = \sinh\chi \ . \tag{22.4}$$

  This describes an *open universe*: the physical separation between objects is totally unbounded.

  For $k = 0$, space has a "flat" Euclidean geometry: for $dt = 0$,

$$ds^2 = a(t)^2\left[d\bar{r}^2 + \bar{r}^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right] \ . \tag{22.5}$$

  This is often called a "flat universe," though that is a bit misleading — space**time** is curved.

22.3 Propagation of light in this spacetime

The value of k and the behavior of $a(t)$ are connected to the matter that fills the universe, and can be determined from the Einstein field equations. Before discussing those quantities, it is useful to examine how light and matter behave in these spacetimes.

Begin by asking what happens to observers at rest in the coordinates: $u^t = c$, $u^{\bar{r}} = u^\theta = u^\phi = 0$. When we examine geodesics, we find that they remain fixed at coordinate $(\bar{r}, \theta, \phi)$. However, as those observers remain fixed at that coordinate, we see that the proper separation of observers changes as $a(t)$ changes. Those observers "co-move" as the universe's geometry changes[3].

[3]Note that this *only* applies to comoving points. This means points which do not experience forces which "push" them away from the geodesic. The separation between us and a very distant galaxy changes as $a(t)$ changes. However, that galaxy's size does not change because it is a *bound object* — it is not built out of things that are comoving in the Robertson-Walker spacetime. The scales of things that are bound together — like stars, planets, solar systems, people — do not change as $a(t)$ changes. The Robertson-Walker spacetime describes the geometry of events on very large scales; it doesn't describe things on small scales.

Next examine light — which is our main tool for measuring and understanding our universe. For simplicity, we will focus on $k = 0$. (The calculation can be generalized to $k = \pm 1$, but the details are a bit messy using the tools of 8.033; we'll just quote the result for these cases.) Imagine that light is emitted at some time t_e, and is received by an observer at some time t_r. It's enough to consider light that moves radially, so we'll put $p^\theta = p^\phi = 0$.

Our goal is to compare the energy of light when it is emitted to the energy when it is received. To do this, we imagine one comoving observer measures the light at emission, and another at reception:

$$E_{\rm emit} = -\vec{p}_{\rm emit} \cdot \vec{u}_{\rm emit} = p^t_{\rm emit} c \; . \tag{22.6}$$

Here we used the fact that the comoving observer has only one non-zero 4-velocity component, which we can write $u_t = -c$. Likewise, we find $E_{\rm rec} = p^t_{\rm rec} c$.

Let's now propagate this light across spacetime as a radial geodesic and see what energy it has at $t = t_r$. We use two rules to propagate the light:

1. It follows a light-like trajectory or null trajectory, so $\vec{p} \cdot \vec{p} = 0$:

$$-(p^t)^2 + a^2(t)(p^{\bar{r}})^2 = 0 \quad \rightarrow \quad p^{\bar{r}} = p^t/a(t) \; . \tag{22.7}$$

2. It follows a geodesic, so we extremize

$$L = \frac{1}{2} g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = -\frac{c^2}{2} \left(\frac{dt}{d\lambda}\right)^2 + \frac{a(t)^2}{2} \left(\frac{d\bar{r}}{d\lambda}\right)^2 \; . \tag{22.8}$$

Let's focus on the $x^0 = ct$ component of the Euler-Lagrange equations:

$$\frac{\partial L}{\partial x^0} = \frac{1}{c}\frac{\partial L}{\partial t} = \frac{1}{c} a\dot{a} \left(p^{\bar{r}}\right)^2 \; , \quad \text{where} \quad \dot{a} = \frac{da}{dt} \; ; \tag{22.9}$$

$$\frac{\partial L}{\partial (dx^0/d\lambda)} = -c\frac{dt}{d\lambda} = -p^t \; ; \tag{22.10}$$

$$\frac{d}{d\lambda}\left[\frac{\partial L}{\partial (dx^0/d\lambda)}\right] = -\frac{dp^t}{d\lambda} \; . \tag{22.11}$$

Put all these ingredients together:

$$\frac{\partial L}{\partial x^0} - \frac{d}{d\lambda}\left[\frac{\partial L}{\partial (dx^0/d\lambda)}\right] = 0 \tag{22.12}$$

becomes

$$\frac{a\dot{a}}{c} \left(p^{\bar{r}}\right)^2 + \frac{dp^t}{d\lambda} = 0 \; . \tag{22.13}$$

Using the constraint $\vec{p} \cdot \vec{p} = 0$, this becomes

$$\frac{1}{c}\frac{\dot{a}}{a} \left(p^t\right)^2 + \frac{dp^t}{d\lambda} = 0 \; . \tag{22.14}$$

But we also know that

$$\dot{a}\, p^t = \frac{da}{dt}\, c\, \frac{dt}{d\lambda} = c\, \frac{da}{d\lambda} \; . \tag{22.15}$$

With this, our equation becomes

$$\frac{da/d\lambda}{a}p^t + \frac{dp^t}{d\lambda} = 0\ , \tag{22.16}$$

or

$$\frac{da/d\lambda}{a} = -\frac{dp^t/d\lambda}{p^t}\ , \tag{22.17}$$

Integrate both sides from $\lambda = \lambda_e$ (corresponding to the moment t_e when light is emitted) to $\lambda = \lambda_r$ (corresponding to the moment t_r when light is received):

$$\ln\left[\frac{p^t(t_r)}{p^t(t_e)}\right] = -\ln\left[\frac{a(t_r)}{a(t_e)}\right]\ , \tag{22.18}$$

or

$$\frac{p^t(t_r)}{p^t(t_e)} = \frac{a(t_e)}{a(t_r)}\ . \tag{22.19}$$

From the fact that $E_{\rm emit} = cp^t(t_e)$ and $E_{\rm rec} = cp^t(t_r)$, this means

$$E_{\rm rec} = E_{\rm emit}\left(\frac{a(t_e)}{a(t_r)}\right)\ . \tag{22.20}$$

In other words, *the energy associated with the light that we measure gives us a way to directly probe the scale factor of the universe.* (The result turns out to be identical for $k = \pm 1$.)

So how do we use this? We take advantage of the fact that atoms and molecules whose electrons are in an excited state emit light with distinct spectral lines. Figure 1 illustrates what the spectrum from a gas cloud might look like if the atoms and molecules in the gas all undergo known electronic transitions. The blue curve in this figure illustrates the spectrum in the "rest frame," i.e., what we might measure in a laboratory. In this sketch, we imagine that there are 4 different "lines," each at a wavelength $\lambda_{1,2,3,4}$ that has been very well characterized (e.g., by laboratory measurements and/or theoretical calculations).

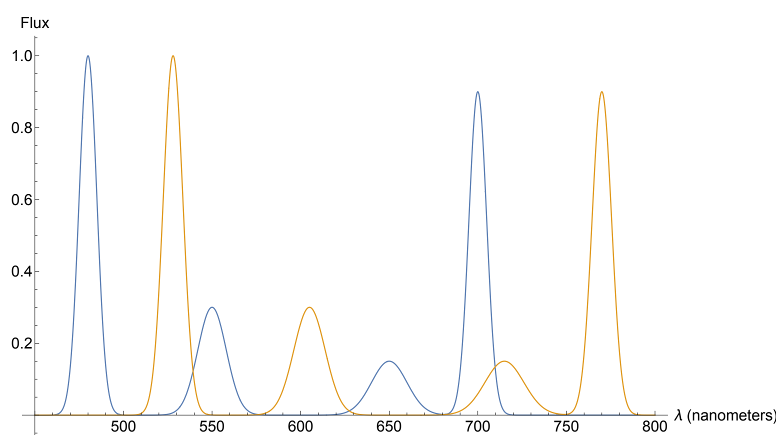

Figure 1: Sketch of how the universe's scale factor affects an object's light emission spectrum. Blue curve sketches a spectrum as it would be viewed in that object's "rest frame." This is what we would measure in the laboratory, for example. Imagine that this light is emitted at t_e, and is measured at t_r. Orange curve shows that same spectrum if it is measured at t_r such that $a(t_r)/a(t_e) = 1.1$.

Imagine that the light is emitted at t_e, when the universe's scale factor is $a(t_e)$. The orange curve in Fig. 1 illustrates what this spectrum might look like if it is measured at t_r, when the scale is now $a(t_r)$. Each photon that contributes to the light has been *redshifted* by the expansion of the universe. Because the energy of light relates to its wavelength according to $E = hc/\lambda$, each "line" at λ_i has been shifted to

$$\lambda_i' = \lambda_i \left(\frac{a(t_r)}{a(t_e)} \right) \equiv \lambda_i (1 + z) . \tag{22.21}$$

This equation defines the *cosmological redshift*, z. This is what we determine when we measure a spectrum and deduce the nature of the atoms or molecules that emitted its light.

The punchline is that by measuring the spectra of distant objects and looking for the "fingerprints" of known[4] atomic and molecular transitions, we can deduce the scale factor at which the light was emitted, compared to the scale factor's value today. If you do this for a large number of sources, you can build up map of how the scale factor evolves. If we understand how the scale factor evolves as a function of time, we can then use measurements of many different sources' redshifts in order to learn how the universe is evolving.

22.4 The behavior of $a(t)$ and k

If you run the Robertson-Walker line element through the Einstein field equation, you find that the scale factor $a(t)$ and the curvature parameter k are related to the energy density of "stuff" in the universe according to

$$\left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi G\rho}{3c^2} - \frac{kc^2}{a^2} . \tag{22.22}$$

This relationship was first discovered by Alexander Friedmann in 1922, and is known as the Friedmann equation. Any Robertson-Walker spacetime for which $a(t)$ and k connect to ρ by this relationship is known as a Friedmann-Robertson-Walker (or FRW) cosmology.

Before discussing some details, it is useful to introduce some terminology:

$$\frac{\dot{a}}{a} \equiv H \qquad \text{The "Hubble" expansion parameter.} \tag{22.23}$$

Noice that this parameter has the dimensions of 1/time. The value of the Hubble parameter today is a subject of quite a bit of active research:

$$H_0 \equiv H(t = \text{now}) \approx 70(\text{km/sec})/\text{Mpc} . \tag{22.24}$$

The precise value of H_0 is somewhat controversial as I write this document, with different techniques yielding somewhat different values, ranging from about 67 in these units up to about 73. Not that long ago, methods that yielded values this close to one another (each differs by about 5% from 70) would have been celebrated as a triumph; when I started graduate school, people were concerned about whether the value was closer to 50 or to 100. One reason that there is a lot of interest in the different values obtained by current measurements is that it is not clear whether these numbers reflect different systematic uncertainties

[4]Note that in principle there's a big assumption being used here: We assume that the basic physics describing atoms and molecules is the same now as when and where the light was emitted.

in the different methods, or whether the physics of the different methods means that they are measuring fundamentally different things.

Another useful parameter is a critical density:

$$\rho_{\rm crit} = \frac{3H^2c^2}{8\pi G} \ . \tag{22.25}$$

We can normalize density to this value:

$$\Omega \equiv \frac{\rho}{\rho_{\rm crit}} \ , \tag{22.26}$$

and then rearrange the Friedmann equation using this definition:

$$1 = \frac{8\pi G\rho}{3H^2c^2} - \frac{kc^2}{a^2H^2} = \frac{\rho}{\rho_{\rm crit}} - \frac{kc^2}{a^2H^2} \ , \tag{22.27}$$

or

$$\Omega - 1 = \frac{kc^2}{a^2H^2} \ . \tag{22.28}$$

This lets us see the significance of $\rho_{\rm crit}$:

- If $\rho > \rho_{\rm crit}$, then $\Omega > 1$ and we must have k positive. *We must have a spatially closed universe if $\rho > \rho_{\rm crit}$.* It can be shown in this case that the universe expands to a maximum size, then recollapses.
- If $\rho < \rho_{\rm crit}$, then $\Omega < 1$ and we must have k negative. *We must have a spatially open universe if $\rho < \rho_{\rm crit}$.* It can be shown in this case that the universe expands forever.
- If $\rho = \rho_{\rm crit}$, then $\Omega = 1$ and we must have $k = 0$. *We must have a spatially flat universe if $\rho = \rho_{\rm crit}$.* It can be shown in this case that the universe expands forever, but (in most cases) with ever decreasing speed. (There is one interesting an important exception to this trend, which we describe in more detail below.)

To know which of these options corresponds to our universe, we need to know how the universe behaves depending on the mixture of "stuff" that goes into it. This is in general a complicated problem, but we can get insight by looking at a couple of illustrative limiting cases. Let's take a universe with $k = 0$ and fill it with matter in the form of dust[5]. In this limit, the total number of dust particles is fixed, but their density changes as $a(t)$ changes:

$$\rho_M(t) = \rho_M({\rm now})\left[\frac{a({\rm now})}{a(t)}\right]^3 \equiv \rho_0\frac{a_0^3}{a(t)^3} \ . \tag{22.29}$$

When you plug this in to the Friedmann equation (with $k = 0$), you find that $a(t)$ can be solved using a power-law in time:

$$a(t) = a_0\left(\frac{t}{t_0}\right)^n \ . \tag{22.30}$$

Running this through Eq. (22.22), we find $n = 2/3$. This tells us that in a spatially flat "matter-dominated" universe, the scale factor grows as a function of time as $a \propto t^{2/3}$.

[5]Recall that "dust" can be thought of as a perfect fluid with $P = 0$.

This solution implies an expanding universe. If you run it backwards, it implies that $a = 0$ at some point in the past. This means that all spatial locations were smashed into a single zero-size point (assuming that the FRW model holds all the way back to that moment — perhaps a rather big assumption!). Spacetime itself comes into existence as we evolve from that moment. The birth of all of space is known as the "Big Bang." Notice it is not an explosion *into* space — it is the creation of space itself. There wasn't any "there" to explode into until the Big Bang happened!

Another representative example: a universe filled with radiation. Imagine that the number of photons is fixed, but their density varies as $a(t)^{-3}$. In addition, each photon has an energy that itself varies as $1/a(t)$ — the redshift effect. This implies that the energy density of radiation obeys

$$\rho_R(t) = \rho_0 \left(\frac{a_0}{a(t)}\right)^4 . \tag{22.31}$$

This also admits a power-law solution; running it through Friedmann, we find $a(t) \propto t^{1/2}$ in a "radiation-dominated" universe.

One last example has been found to be very important — "vacuum energy," also known as a "cosmological constant." The vacuum energy arises in quantum field theory as an energy associated with the ground state of quantum fields. Its key property is that it must be invariant with respect to Lorentz transformations in the freely-falling frame: $T^{\mu\nu} \propto \eta^{\mu\nu}$ in the FFF. This means that this variety of "stuff" looks like a perfect fluid, but one with *negative pressure*:

$$P_\Lambda = -\rho_\Lambda . \tag{22.32}$$

We can see how this contribution evolves by enforcing the rule that the stress-energy tensor be divergence free; doing so, we find out that ρ_Λ is *constant* with time. This rather odd behavior is a consequence of the fact that this "fluid" is associated with the vacuum itself.

When we plug this behavior for the density into the Friedmann equation, here's what we get:

$$\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G\rho_\Lambda}{3c^2} \\ \dot{a} &= \pm a\sqrt{\frac{8\pi G\rho_\Lambda}{3c^2}} \\ a(t) &\propto \exp\left[\pm t\sqrt{\frac{8\pi G\rho_\Lambda}{3c^2}}\right] . \end{aligned} \tag{22.33}$$

This solution yields exponential expansion. (Or contraction; however, expansion dominates, since the contracting solution rapidly crushes away its own relevance.) This case is the exception to $k = 0$ describing expansion with ever decreasing speed. Exponential expansion for $a(t)$ accelerates with time.

The three cases discussed here — matter-dominated, radiation-dominated, vacuum-energy-dominated — are idealized, but demonstrate how different contributions to the universe give different ways in which $a(t)$ evolves with time. We generally expect a mixture of different ingredients, for which these power-law solutions don't apply. But, these limits provide asymptotic solutions which are useful for guiding our understanding. The general case is not too hard to solve for by integrating the Friedmann equation numerically. By measuring the rate of expansion at many different times and comparing to different models, we can infer what our universe is (apparently) made of.

22.5 Measurements and our universe

What we really want, then, is to measure a (which is encoded in the redshift of distant sources) at many values of t. This will let us build up $a(t)$; connect this to some good models for matter in the universe, and we should be able to learn something interesting.

Our main tool for doing this is to measure the *distance* to different objects. Since light's travel speed is known, distance tells us the time at which light was emitted. Time or distance plus redshift lets us build $a(t)$. Two tools are particularly important for doing this:

- *Standard rulers* are sources whose size is known by some physics. We compare the apparent size to the physical size; the ratio tells us the source's distance.
- *Standard candles* are sources whose intrinsic brightness is known. Compare the apparent and intrinsic brightness; the ratio again tells us the source's distance.

Doing measurements of this kind is an industry. The basic idea is to build a large data set containing high-quality data describing distance versus redshift for class of sources, and then find the solution to the Friedmann equations — some self-consistent solution with H_0, Ω_M, Ω_Λ, Ω_r, and k — that bests describes these data.

Nearly current data[6] (at least, as of the writing of these notes) tells us

$$\Omega_M = \rho_M/\rho_{\rm crit} = 0.311 \pm 0.006 \tag{22.34}$$

$$\Omega_\Lambda = \rho_\Lambda/\rho_{\rm crit} = 0.689 \pm 0.006 \tag{22.35}$$

$$\Omega_{\rm total} \equiv \Omega_M + \Omega_\Lambda = 0.9993 \pm 0.0019\ . \tag{22.36}$$

(The contribution of radiation, Ω_r, is so small it doesn't show up in this table.) The data are consistent with $k = 0$, telling us that our universe appears to be spatially flat.

This is lovely ... but there is some weirdness under the hood. Here are a few current mysteries:

1. **What's the real value of H_0?** As mentioned, the value of H_0 is something that different techniques disagree on. The table above is based on one of those values (which is "self consistent" with the technique that contributes the most to that dataset), but other values differ. Is there something prosaic skewing some of the measurements? Or is there something deeper going on — perhaps we have overlooked some contributor to the Friedmann equations whose importance is not obvious right now?

2. **Why is $k = 0$?** One can show that if $\Omega - 1 = \epsilon$, then $|\epsilon|$ *grows* with time in a matter- or radiation-dominated universe (becoming larger in magnitude, whether positive or negative). In other words, the deviation from spatial flatness should be magnified as the universe evolves, if the universe is matter or radiation dominated. Observations indeed indicate that our universe is matter dominated now, and was radiation dominated long ago (greater than about 13.5 billion years ago). For ϵ to be so close to zero today, it would have had to be even closer — many digits closer — at a very early time in the universe's history.

   If, however, the universe is *not* matter or radiation dominated, but is instead vacuum-energy dominated, then it is not hard to show that $\Omega - 1$ evolves to zero as $a(t)$

[6] Numbers taken from `https://pdg.lbl.gov/2021/reviews/rpp2021-rev-cosmological-parameters.pdf`

exponentially expands. A way out is thus to imagine that the universe was in such a state at very early times — perhaps very, very early in the universe's history, before it became radiation dominated. The idea that our universe behaved this way constitutes the theory of *cosmic inflation*.

Inflation comes in different flavors, depending upon details of how one designs the energy of the "vacuum" (more correctly, the *false vacuum*) that drives the expansion. The version most people look at for this today, whose foundations were developed by Alan Guth about 40 years ago, suggests that our universe exponentially inflated for about 10^{-30} seconds at a very early time. If this is the case, then inflation very likely left a mark in the form of very weak gravitational waves that have a unique and very broad spectrum, stretching from the band to which LIGO is sensitive now, down to frequencies of order 1/(billions of years). Searching for the imprint of these waves is one of the top problems in observational cosmology today.

3. **What is the matter that contributes to Ω_M?** If we add up all the matter we can see that produces light — stuff we know about from the standard model of particle physics — we get

$$\Omega_b = 0.0489 \pm 0.0003 \; . \tag{22.37}$$

(The b on this symbol stands for "baryon," since most of the mass comes from protons and neutrons and the atoms that are built from them.) This is *way* smaller than $\Omega_M = 0.311$. The remaining $\Omega_{\rm DM} = 0.262$ is apparently some kind of "dark" matter. We can see its gravitational influence, but have never detected any "dark matter particle" in any experiment. Lots of people have proposed different ways that matter can produce gravity, but (apparently!) not couple to electromagnetic fields (or, at best, couple weakly enough to evade all detection limits so far). We're still working on this one.

4. **What is Ω_Λ?** The fact that the vacuum energy plays an important role in cosmology today was a rather large surprise when it was first clearly measured about 25 years ago. We are kind of baffled as to what this ingredient in the universe's "energy budget" consists of; indeed, just last year, preliminary evidence was presented hinting that it might not be the "cosmological constant" that one normally thinks of in this context, but might be something even weirder[7].

It is very interesting that when we apply general relativity to compact, strong-gravity objects, it passes every quantitative test we have been able formulate so far. When we apply general relativity on the largest scales, we find it can describe what we observe just fine, but it tells us that our universe is even weirder than we realized. This is a story which is not even close to being over.

[7]See https://arXiv.org/abs/2404.08056 for references presenting this preliminary evidence, as well as discussion urging caution about the "evolving dark energy" interpretation.