Reading a distant clock in a Robertson-Walker spacetime

During some of the Zoom lectures on cosmology, the question has come up as to how we know that a cosmologically distant clock is seen by us to "run slow." We all seem to "know" that an interval of time Δt on a distant clock should be measured by us here on Earth as an interval $\Delta t_z = (1+z)\Delta t$. How can we show this?

To begin, let us write the Robertson-Walker line element as follows:

$$ds^2 = -dt^2 + a^2(t)R_0^2\left[d\chi^2 + S_k(\chi)d\Omega^2\right] .$$

Note that t is proper time for *all* comoving observers in this spacetime: in these coordinates, the 4-velocity of a comoving observer has components $u^\alpha \doteq (dt/d\tau, d\chi/d\tau, d\theta/d\tau, d\phi/d\tau) = (1,0,0,0)$. During some of our Zoom discussions, I may have said something about the nature of the time coordinate that contradicted this. If so, I was incorrect. In fact, whenever a metric has $g_{tt} = -1$ and $g_{ti} = 0$, the coordinate t is the proper time of a comoving observer in that spacetime.

Next, imagine a clock at radial coordinate χ_e. Photons are emitted from the clock at $t = t_{e1}$ and at $t = t_{e2}$, carrying a picture of the clock's face at these moments. The emission times of these photons bound a time interval $\Delta t_e = t_{e2} - t_{e1}$. The photons then propagate radially inward to $\chi = 0$, and are measured at times t_{m1} and t_{m2}. Our goal is to compute the time interval $\Delta t_m = t_{m2} - t_{m1}$ measured at $\chi = 0$.

Both photons travel along null geodesics for which $0 = -dt^2 + a^2(t)R_0^2\,d\chi^2$, or

$$R_0\,d\chi = -dt/a(t) .$$

(Choosing the minus sign for the inward radial trajectory.) For the first photon, we have

$$0 - R_0\chi_e = -\int_{t_{e1}}^{t_{m1}} \frac{dt}{a(t)} .$$

For the second photon,

$$0 - R_0\chi_e = -\int_{t_{e2}}^{t_{m2}} \frac{dt}{a(t)} .$$

Since χ is a comoving radial coordinate, both of these photons are emitted from $\chi = \chi_e$ and are measured at the same $\chi = 0$. The left-hand sides of these equations are thus identical. We can therefore equate their right-hand sides to each other and rearrange:

$$\begin{aligned} 0 &= \int_{t_{e2}}^{t_{m2}} \frac{dt}{a(t)} - \int_{t_{e1}}^{t_{m1}} \frac{dt}{a(t)} \\ &= \int_{t_{m1}}^{t_{m2}} \frac{dt}{a(t)} - \int_{t_{e1}}^{t_{e2}} \frac{dt}{a(t)} . \end{aligned}$$

On the second line, we've used the fundamental theorem of calculus to rearrange the limits of integration. If we assume that both Δt_e and Δt_m are short compared to the timescale over which the scale factor $a(t)$ changes, then we can write

$$0 = \frac{\Delta t_m}{a(t_m)} - \frac{\Delta t_e}{a(t_e)}$$

[using $a(t_e) \equiv a(t_{e1}) \simeq a(t_{e2})$, and likewise for $a(t_m)$]. Taking the time of measurement to be now, so that $a(t_m) = 1$, we find

$$\Delta t_m = \frac{\Delta t_e}{a(t_e)} \equiv [1 + z(t_e)]\Delta t_e .$$

We measure a time interval Δt_e on a cosmologically distant clock redshifted to $\Delta t_m = (1+z)\Delta t_e$.