Gauge transformations on a curved background

This notes accompany the discussion of the "generalized gauge transformation" given in Lecture 17, page 4. Begin with a spacetime whose metric we represent as some background $\hat{g}_{\alpha\beta}$ plus a perturbation $h_{\alpha\beta}$; these are all expressed in terms of the coordinate x^γ. We wish to change to a new coordinate y^γ which is related to x^γ by an infinitesimal shift ξ^γ:

$$x^\alpha = y^\alpha - \xi^\alpha(y^\gamma)\ .$$

This coordinate transformation changes the metric according to

$$g_{\alpha\beta}(y^\gamma) \;=\; g_{\gamma\delta}(x^\gamma)\frac{\partial x^\gamma}{\partial y^\alpha}\frac{\partial x^\delta}{\partial y^\beta}$$

Let's now expand this out:

$$\begin{aligned}
g_{\alpha\beta}(y^\gamma) &= (\hat{g}_{\gamma\delta} + h_{\gamma\delta})\,(\delta^\gamma{}_\alpha - \partial_\alpha\xi^\gamma)\left(\delta^\delta{}_\beta - \partial_\beta\xi^\delta\right) \\
&= \hat{g}_{\alpha\beta} + h_{\alpha\beta} - \hat{g}_{\gamma\beta}\partial_\alpha\xi^\delta - \hat{g}_{\alpha\delta}\partial_\beta\xi^\gamma + \mathcal{O}[h\,\partial\xi, (\partial\xi)^2] \\
&= \hat{g}_{\alpha\beta} + h_{\alpha\beta} - \hat{g}_{\gamma\beta}(\hat{\nabla}_\alpha\xi^\gamma - \xi^\delta\hat{\Gamma}^\gamma_{\alpha\delta}) - \hat{g}_{\alpha\delta}(\hat{\nabla}_\beta\xi^\delta - \xi^\gamma\hat{\Gamma}^\delta_{\beta\gamma}) \\
&= \hat{g}_{\alpha\beta} + h_{\alpha\beta} - \hat{\nabla}_\alpha\xi_\beta - \hat{\nabla}_\beta\xi_\alpha + \xi^\gamma(\hat{\Gamma}_{\beta\alpha\gamma} + \hat{\Gamma}_{\alpha\beta\gamma}) \\
&= \hat{g}_{\alpha\beta} + h_{\alpha\beta} - \hat{\nabla}_\alpha\xi_\beta - \hat{\nabla}_\beta\xi_\alpha + \xi^\gamma\partial_\gamma\hat{g}_{\alpha\beta}\ .
\end{aligned}$$

On going from the fourth to the fifth line, we've used the definition of the Christoffel symbol. On that last line, the background in this expression is currently expressed as a function of the old coordinates, x^γ. We want it to appear as a function of the new coordinates, y^γ:

$$\begin{aligned}
\hat{g}_{\alpha\beta}(x^\gamma) &= \hat{g}_{\alpha\beta}(y^\gamma - \xi^\gamma) \\
&= \hat{g}_{\alpha\beta}(y^\gamma) - \xi^\gamma\partial_\gamma\hat{g}_{\alpha\beta}\ .
\end{aligned}$$

Note, we could do this expansion for the perturbation as well: $h_{\alpha\beta}(x^\gamma) = h_{\alpha\beta}(y^\gamma - \xi^\gamma) = h_{\alpha\beta}(y^\gamma) - \xi^\gamma\partial_\gamma h_{\alpha\beta}$. This means that $h_{\alpha\beta}(x^\gamma) = h_{\alpha\beta}(y^\gamma) +$ terms that are of order "small squared," which we neglect.

Substituting the result for $\hat{g}_{\alpha\beta}(x^\gamma)$ into $g_{\alpha\beta}(y^\gamma)$ yields

$$g_{\alpha\beta}(y^\gamma) \;=\; \hat{g}_{\alpha\beta} + h_{\alpha\beta} - \hat{\nabla}_\alpha\xi_\beta - \hat{\nabla}_\beta\xi_\alpha\ ,$$

from which we read off

$$h_{\alpha\beta} \to h_{\alpha\beta} - \hat{\nabla}_\alpha\xi_\beta - \hat{\nabla}_\beta\xi_\alpha\ .$$