More on geodesic deviation

During one of the Zoom lectures, I was asked about a step in the calculation of geodesic deviation in Lecture 11, particularly the boundary condition on the derivative of the separation vector shown at the bottom of page 4 of the notes for this lecture. As I examined how to answer this question, I realized that the whole thing could use a bit of cleanup. My intention in the original presentation of these notes was for certain terms to be written down exactly, but then for the bulk of the calculation to be done in a local Lorentz frame (LLF) centered on a particular event.

These notes revisit this entire calculation. I first present geodesic deviation in a very formal way that requires no particular assumptions about the frame in which the calculation is done. I then revisit the "centered on the LLF" calculation, re-doing the calculation in Lecture 11. The formal calculation is entirely rigorous, making no approximations or particular assumptions. The second calculation should build in more physical insight. This version of the calculation also demonstrates how one can do a computation in a particular reference frame, but then express it in a tensorial way. The principle that "tensorial in one frame means tensorial in all frames" then allows us to infer that the resulting expression is good in general. (Plus, in this case, what we find this way is identical to what we find with the formal calculation.)

First, the formal calculation. Consider a two-parameter family of geodesic trajectories in spacetime $x^\alpha(\lambda, s)$. The parameter λ is affine parameter along each geodesic; hence $u^\alpha(s) = (\partial x^\alpha/\partial\lambda)_s$ is the tangent along the geodesic with parameter s. (If we choose λ to be proper time, then u^α is the usual 4-velocity.)

The parameter s tells us about the separation of neighboring geodesics. The vector[1] $Y^\alpha(\lambda) = (\partial x^\alpha/\partial s)_\lambda$ points from the event at affine parameter λ on the geodesic at s to the event at affine parameter λ on the geodesic at $s + ds$.

Note that

$$u^\beta\nabla_\beta Y^\mu = \frac{\partial^2 x^\mu}{\partial\lambda\partial s} + \Gamma^\mu{}_{\beta\nu}u^\beta Y^\nu \quad \text{and} \quad Y^\beta\nabla_\beta u^\mu = \frac{\partial^2 x^\mu}{\partial s\partial\lambda} + \Gamma^\mu{}_{\beta\nu}Y^\beta u^\nu \; .$$

Partial derivatives commute, the connection is symmetric on downstairs indices 2 and 3, and we can relabel dummy indices, so we find $u^\beta\nabla_\beta Y^\mu = Y^\beta\nabla_\beta u^\mu$.

Let us now evaluate the *covariant acceleration* of the separation vector Y^μ:

$$\begin{aligned}
\frac{D^2Y^\mu}{d\lambda^2} &\equiv u^\alpha\nabla_\alpha\left(u^\beta\nabla_\beta Y^\mu\right)\\
&= u^\alpha\nabla_\alpha\left(Y^\beta\nabla_\beta u^\mu\right)\\
&= \left(u^\alpha\nabla_\alpha Y^\beta\right)\left(\nabla_\beta u^\mu\right) + u^\alpha Y^\beta\nabla_\alpha\nabla_\beta u^\mu\\
&= \left(u^\alpha\nabla_\alpha Y^\beta\right)\left(\nabla_\beta u^\mu\right) + u^\alpha Y^\beta\left(\nabla_\beta\nabla_\alpha u^\mu + R^\mu{}_{\nu\alpha\beta}u^\nu\right)\\
&= \left(u^\alpha\nabla_\alpha Y^\beta\right)\left(\nabla_\beta u^\mu\right) + Y^\beta\nabla_\beta\left(u^\alpha\nabla_\alpha u^\mu\right) - \left(Y^\beta\nabla_\beta u^\alpha\right)\left(\nabla_\alpha u^\mu\right) + R^\mu{}_{\nu\alpha\beta}u^\nu u^\alpha Y^\beta\\
&= R^\mu{}_{\nu\alpha\beta}u^\nu u^\alpha Y^\beta \; .
\end{aligned}$$

The first line is essentially a definition. To get to the second line, we use the identity shown above. To go to the third line, we expand the action of ∇_α. To go to the fourth line, we use the commutator rule $[\nabla_\alpha, \nabla_\beta]u^\mu = R^\mu{}_{\nu\alpha\beta}u^\nu$.

In the next step, we use $u^\alpha\left(\nabla_\beta W_\alpha{}^\mu\right) = \nabla_\beta\left(u^\alpha W_\alpha{}^\mu\right) - \left(\nabla_\beta u^\alpha\right)W_\alpha{}^\mu$ (with $W_\alpha{}^\mu = \nabla_\alpha u^\mu$). To simplify the fifth line, we use the fact that the first and third terms on the right-hand side are equal but opposite in sign after using the identity and relabeling dummy indices, as well as the fact that the second term vanishes since u^μ is the tangent along a geodesic trajectory.

The final line gives us the covariant acceleration of the separation vector, and is the equation of geodesic deviation.

[1] This was labeled ξ^α in Lecture 11. I am changing here because $\vec{\xi}$ also denotes a Killing vector, and ξ^α used in the lecture on linearized gravity to describe the generator of an infinitesimal coordinate transformation. The letter ξ is a bit overloaded!

Next, a calculation that focuses on the LLF. Consider two nearby geodesics, both parameterized by affine parameter λ: Geodesic 1 follows the curve $x^\alpha(\lambda)$, and has tangent vector $u^\alpha = dx^\alpha/d\lambda$; geodesic 2 follows the curve $z^\alpha(\lambda)$, and has tangent vector $v^\alpha = dz^\alpha/d\lambda$. Let $Y^\alpha = z^\alpha - x^\alpha$ be the vector that points from events at parameter λ on geodesic 1 to events at λ on geodesic 2.

We expand all important elements of the geometry in a LLF centered on event A at $\lambda = \lambda_0$ on geodesic 1. Let A' be the corresponding event on geodesic 2. We assume that A and A' are sufficiently close to one another that their tangent vectors are parallel at $\lambda = \lambda_0$; we elaborate on what this means quantitatively below. Other important quantities we need are the metric and the connection at these two events:

$$g_{\mu\nu}\big|_A = \eta_{\mu\nu}\ , \qquad g_{\mu\nu}\big|_{A'} = \eta_{\mu\nu}\ ; \qquad \Gamma^\mu{}_{\alpha\beta}\big|_A = 0\ , \qquad \Gamma^\mu{}_{\alpha\beta}\big|_{A'} = Y^\gamma\left(\partial_\gamma\Gamma^\mu{}_{\alpha\beta}\right)\big|_A\ .$$

The geodesic equation for geodesic 1, evaluated at event A, is

$$\left.\frac{d^2x^\mu}{d\lambda^2}\right|_A = 0\ .$$

The corresponding equation for geodesic 2, evaluated at event A', is

$$\left.\frac{d^2z^\mu}{d\lambda^2}\right|_{A'} + \left(\Gamma^\mu{}_{\alpha\beta}v^\alpha v^\beta\right)\big|_{A'} = 0\ .$$

Combining different definitions and assumptions, this can be rewritten

$$\left.\frac{d^2z^\mu}{d\lambda^2}\right|_{A'} = -Y^\gamma\left(\partial_\gamma\Gamma^\mu{}_{\alpha\beta}u^\alpha u^\beta\right)\big|_A\ . \tag{1}$$

Here we used $v^\alpha = u^\alpha$, which follows from our requirement that the tangents to geodesics 1 and 2 be parallel at $\lambda = \lambda_0$. This requires us to parallel transport from A' to A, but this transport is trivial[2] in the LLF.

Taking the difference between the geodesic equation on geodesic 2 and on geodesic 1, we find

$$\frac{d^2Y^\mu}{d\lambda^2} = -\partial_\gamma\Gamma^\mu{}_{\alpha\beta}u^\alpha u^\beta Y^\gamma\ .$$

We drop the A and A' subscripts from here on; all quantities are now evaluated at A.

This acceleration equation uses $d/d\lambda = u^\alpha\partial_\alpha$ as its derivative operator. To write this in a way that nicely translates into proper tensorial form, we need to express things in terms of $D/d\lambda \equiv u^\alpha\nabla_\alpha$. Begin by looking at the "velocity" equation for the separation vector:

$$\frac{DY^\mu}{d\lambda} = u^\alpha\nabla_\alpha Y^\mu = \frac{dY^\mu}{d\lambda} + \Gamma^\mu{}_{\alpha\beta}Y^\alpha u^\beta\ .$$

You may be tempted to set the connection to zero since we are in the LLF. Wait until we have taken our next derivative before doing that — the fact that the connection has non-zero slope in the LLF is important.

$$\begin{aligned}
\frac{D^2Y^\mu}{d\lambda^2} &= u^\gamma\nabla_\gamma\left(\frac{dY^\mu}{d\lambda} + \Gamma^\mu{}_{\alpha\beta}Y^\alpha u^\beta\right)\\
&= \frac{d^2Y^\mu}{d\lambda^2} + u^\gamma\Gamma^\mu{}_{\gamma\nu}\frac{dY^\nu}{d\lambda} + \left(u^\gamma\nabla_\gamma\Gamma^\mu{}_{\alpha\beta}\right)u^\beta Y^\alpha + \Gamma^\mu{}_{\gamma\nu}\left(u^\gamma\nabla_\gamma u^\beta\right)Y^\alpha + \Gamma^\mu{}_{\alpha\beta}u^\beta\left(u^\gamma\nabla_\gamma Y^\mu\right)\\
&= \frac{d^2Y^\mu}{d\lambda^2} + \partial_\gamma\Gamma^\mu{}_{\alpha\beta}u^\beta u^\gamma Y^\alpha + O(\Gamma^2)\ .
\end{aligned}$$

On the final line of this equation, we have gone into the LLF near event A, we have used the fact that u^α is a geodesic, and we used the fact that $dY^\mu/d\lambda = v^\mu - u^\mu = 0$.

Finally, use the previous results we found for $d^2Y^\mu/d\lambda^2$ to find

$$\begin{aligned}
\frac{D^2Y^\mu}{d\lambda^2} &= \partial_\gamma\Gamma^\mu{}_{\alpha\beta}u^\beta u^\gamma Y^\alpha - \partial_\gamma\Gamma^\mu{}_{\alpha\beta}u^\alpha u^\beta Y^\gamma\\
&= \left(\partial_\gamma\Gamma^\mu{}_{\alpha\beta} - \partial_\alpha\Gamma^\mu{}_{\beta\gamma}\right)u^\beta u^\gamma Y^\alpha\\
&= R^\mu{}_{\beta\gamma\alpha}u^\beta u^\gamma Y^\alpha\ .
\end{aligned}$$

[2] At least to linear order in important quantities; if you do this transport and keep all terms, you will find an additional term of order Γ^2. We discard this term at the end of the calculation by our assumption that we are in the LLF.

In going from the first to the second line, we relabeled dummy indices; to go to the third line, we identified that combination of connection derivatives as the Riemann curvature in the LLF. The final result is identical to that found in the formal calculation, modulo a few different choices of dummy index labels.

This second version of the calculation highlights how it is that the second order terms in the metric (via the first order correction to the connection) cause nearby geodesics to deviate from one another. This is the hallmark of a *tidal* effect: the geodesics deviate because “free fall” means slightly different things to freely falling observers that are slightly separated from one another.