Course info & syllabus: See stellar link that is emailed to the class

Text: Carroll, required

Schutz, MTW - good supplements

Organization: Lecturer Hughes, 37-602A sahughes@mit.edu

TAs Ryan Weller & Cagin Yunus

11 psets: 10 worth 9%, 1 worth 10%
Due at beginning of Thursday class, assigned the previous Thursday

Recitations: Mon 4pm, Fri 11am - run like office hours

Class is 50% formalism, tools, basic ideas; 50% applications, many drawn from astrophysics

1st few weeks: The physics of special relativity presented in a way that emphasizes its geometrical nature. Lets us introduce tools, notation, formalism in a way that carries over to general relativity with ease.

Spacetime: A manifold of events endowed with a metric.

Manifold: For our purposes, a set of points with well-understood connectedness properties, and "nice" smoothness properties. No notion of geometry!

→ Much more careful & vigorous discussion given in Carroll, pp 54-62.

Much of the rigor & machinery Carroll develops not needed for this class.

Event: When & where something happens. Labeled with coordinates ... but is a geometric object with meaning totally independent of the coordinate representation!

Metric: a notion of distance between events in spacetime. Such a notion must exist for physics to work! However, it is independent of the concept of manifold - a manifold does not itself include notions of distance between constituent points.

→ Puts geometry in manifold.

Will make rigorous later.

Special relativity: Simplest theory of spacetime; turns out to correspond to zero gravity limit of general relativity.

KEY CONCEPT: Inertial Reference Frame. Define very carefully, following Blandford & Thorne:

Frame is a (conceptual) lattice of clocks & measuring rods that allows us to label - assign coordinates to - spacetime events.

Properties: (i) Lattice moves freely through spacetime - no forces act on it. Also does not rotate relative to distant beacons - no "non-inertial" forces.

(ii) Measuring rods are orthogonal & uniformly ticked, forming an orthonormal, Cartesian coordinate system.

(iii) Clocks tick uniformly.

(iv) Clocks synchronized using "Einstein synchronization": Clock 1 emits light, bounces it off a mirror on clock 2, receives it back. Set clock 2 such that

$$t_{\text{Bounce}} = \frac{1}{2}\left(t_{\text{emit}} + t_{\text{receive}}\right)$$

An event is represented in this IRF by the coordinates:

$$\mathcal{P} = (t_{\mathcal{P}}, x_{\mathcal{P}}, y_{\mathcal{P}}, z_{\mathcal{P}})$$

→ Segue into units

Let $\mathcal{O}$ be an observer who uses this IRF.

We can now define the displacement vector between 2 events:

$$\mathcal{P} \xrightarrow{\Delta \vec{x}} \mathcal{Q}$$

"Component" of vector

$$\Delta \vec{x} \underset{\mathcal{O}}{\doteq} (t_{\mathcal{Q}} - t_{\mathcal{P}},\ x_{\mathcal{Q}} - x_{\mathcal{P}},\ y_{\mathcal{Q}} - y_{\mathcal{P}},\ z_{\mathcal{Q}} - z_{\mathcal{P}})$$

↳ The vector $\Delta \vec{x}$ is represented by this set of numbers in the coordinate system used by $\mathcal{O}$.

Notation: $\Delta \vec{x} \rightarrow \{\Delta x^\mu\}$ $\quad \mu \in [0, 1, 2, 3]$

note: $x^0 = t$, $x^1 = x$, $x^2 = y$, $x^3 = z$

greek indices label spacetime ~~[illegible]~~ components.

Sometimes only want or need spatial pieces, at least as seen by observer $\mathcal{O}$:

$$\Delta \underset{\sim}{x} \underset{\mathcal{O}}{\doteq} (x_{\mathcal{Q}} - x_{\mathcal{P}},\ y_{\mathcal{Q}} - y_{\mathcal{P}},\ z_{\mathcal{Q}} - z_{\mathcal{P}})$$

$$\rightarrow \{\Delta x^i\} \qquad i \in [1, 2, 3]$$

Latin indices for purely spatial components.

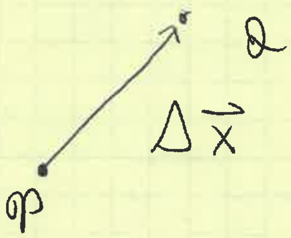
DO THIS FIRST ON THIS PAGE!

Units: choose tickings such that basic unit of length is 1 light second. Then,

$$c = \frac{1 \text{ light-second}}{\text{second}} = 1\,!$$

Speed of light is dimensionless. (Means we scale all speeds to c: $V_{us} = V_{normal} / c_{normal}$.)

A different inertial observer:



Same events, but seen by an observer $\bar{\mathcal{O}}$ who is moving at a speed v along the x direction as seen by $\mathcal{O}$.

Geometric objects $\mathcal{P}$, $\mathcal{Q}$, $\Delta\vec{x}$ are the same, but their representations according to $\bar{\mathcal{O}}$ are not the same as the representations used by $\mathcal{O}$:

$$\Delta\vec{x} \xrightarrow[\bar{\mathcal{O}}]{} \{\Delta x^{\bar{\mu}}\} \qquad \bar{\mu} \in [0,1,2,3]$$

Lorentz transformation relates components as seen by one observer to components as seen by another:

$$\left.\begin{aligned} \Delta x^{\bar{0}} &= \gamma\Delta x^0 - \gamma v \Delta x^1 \\ \Delta x^{\bar{1}} &= -\gamma v \Delta x^0 + \gamma \Delta x^1 \\ \Delta x^{\bar{2}} &= \Delta x^2 \\ \Delta x^{\bar{3}} &= \Delta x^3 \end{aligned}\right\} \text{LT with } c=1,\ \gamma = \frac{1}{\sqrt{1-v^2}}$$

or $$\Delta x^{\bar{\mu}} = \sum_{\nu=0}^{3} \Lambda^{\bar{\mu}}{}_{\nu}\, \Delta x^{\nu} \equiv \Lambda^{\bar{\mu}}{}_{\nu}\, \Delta x^{\nu}$$

$\Lambda^{\bar{\mu}}{}_{\nu}$ = Lorentz transformation matrix. ↳ Einstein summation convention [Sum over repeated indices is assumed.]

Operation is "passive" transformation: Changed the representation of the vector, NOT the vector itself.

Notice: $\Lambda^{\bar{\mu}}{}_{\nu} = \dfrac{\partial x^{\bar{\mu}}}{\partial x^{\nu}}$ → More general form!

Note: irrelevant how we label index ν as long as it is distinct from all other indices in the problem: $\Delta x^{\bar{\mu}} = \Lambda^{\bar{\mu}}{}_{\nu} \Delta x^{\nu} = \Lambda^{\bar{\mu}}{}_{\alpha} \Delta x^{\alpha} = \ldots$

Summed index is a "dummy index." $\bar{\mu}$ is not dummy - sometimes called a "free index."

Definition of a vector: Any quartet of numbers ("components") that transforms between IRFs like the displacement vector: $\vec{A} \underset{\mathcal{O}}{\doteq} (A^0, A^1, A^2, A^3) \underset{\mathcal{O}}{\doteq} \{A^{\alpha}\}$

$$A^{\bar{\mu}} = \Lambda^{\bar{\mu}}{}_{\alpha} A^{\alpha}$$

Key to this definition is the transformation law, which nails down the idea that the vector is a geometric object in the manifold. Also require linearity rules for a vector space:

$$\vec{A} + \vec{B} \underset{\mathcal{O}}{\doteq} \{A^{\alpha} + B^{\alpha}\}$$

$$a\vec{A} \underset{\mathcal{O}}{\doteq} \{a A^{\alpha}\}$$

Basis vectors: In frame $\mathcal{O}$, 4 clearly special vectors are

$$\vec{e}_0 \doteq (1,0,0,0) \qquad \vec{e}_2 \doteq (0,0,1,0)$$
$$\vec{e}_1 \doteq (0,1,0,0) \qquad \vec{e}_3 \doteq (0,0,0,1)$$

Not component. Labeling member of set.

Compact way of writing this: $(\vec{e}_\alpha)^\beta = \delta_\alpha{}^\beta$

Now, $\vec{A} = A^\alpha \vec{e}_\alpha$ Real equals, not just representation!

How do basis vectors transform between frames?

$$\begin{aligned}
A^\alpha \vec{e}_\alpha &= A^{\bar\alpha} \vec{e}_{\bar\alpha} \\
&= (\Lambda^{\bar\alpha}{}_\beta A^\beta)\vec{e}_{\bar\alpha} \\
&= (A^\beta \Lambda^{\bar\alpha}{}_\beta)\vec{e}_{\bar\alpha} && \text{order does not matter!} \\
&= (A^\alpha \Lambda^{\bar\beta}{}_\alpha)\vec{e}_{\bar\beta} && \text{relabel dummy indices}
\end{aligned}$$

$$\rightarrow \quad A^\alpha(\vec{e}_\alpha - \Lambda^{\bar\beta}{}_\alpha \vec{e}_{\bar\beta}) = 0$$

$$\rightarrow \quad \boxed{\vec{e}_\alpha = \Lambda^{\bar\beta}{}_\alpha \vec{e}_{\bar\beta}}$$

Compare to rule for transforming components:

$$\boxed{A^{\bar\beta} = \Lambda^{\bar\beta}{}_\alpha A^\alpha}$$

Quite different! Simple algorithm, though: Just line up indices.

Inverse: Quite simple for S.R. Lorentz transformation, just reverse velocity - only thing that the LT depends on.

$$\vec{e}_\alpha = \Lambda^{\bar{\beta}}{}_\alpha(\underset{\sim}{v})\,\vec{e}_{\bar{\beta}}$$

Clearly, $$\vec{e}_{\bar{\mu}} = \Lambda^{\nu}{}_{\bar{\mu}}(-\underset{\sim}{v})\,\vec{e}_\nu$$

NOTE: $\Lambda^{\bar{\beta}}{}_\alpha(\underset{\sim}{v})$ is the same matrix as $\Lambda^{\nu}{}_{\bar{\mu}}(-\underset{\sim}{v})$, except that the sign of $\underset{\sim}{v}$ is switched. Indices and bars are just a labeling trick. Rule we use is that the argument is the velocity of the IRF for the top index as seen by the bottom index's frame.

Put transformations together:

$$\begin{aligned}\vec{e}_\alpha &= \Lambda^{\bar{\beta}}{}_\alpha(\underset{\sim}{v})\,\vec{e}_{\bar{\beta}} \\ &= \Lambda^{\bar{\beta}}{}_\alpha(\underset{\sim}{v})\left[\Lambda^{\gamma}{}_{\bar{\beta}}(-\underset{\sim}{v})\,\vec{e}_\gamma\right] \\ &= \left[\Lambda^{\bar{\beta}}{}_\alpha(\underset{\sim}{v})\,\Lambda^{\gamma}{}_{\bar{\beta}}(-\underset{\sim}{v})\right]\vec{e}_\gamma\end{aligned}$$

$$\rightarrow \quad \boxed{\delta_\alpha{}^\gamma = \Lambda^{\bar{\beta}}{}_\alpha\,\Lambda^{\gamma}{}_{\bar{\beta}}}$$

Likewise, $$\boxed{\delta_{\bar{\alpha}}{}^{\bar{\gamma}} = \Lambda^{\beta}{}_{\bar{\alpha}}\,\Lambda^{\bar{\gamma}}{}_{\beta}}$$

Scalar product: Familiar result from special relativity, the "interval" $\Delta s^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$ is invariant - the same to all observers.

Notation:
$$\Delta s^2 \equiv \Delta \vec{x} \cdot \Delta \vec{x}$$
$$\underset{\mathcal{O}}{\doteq} -(\Delta x^0)^2 + (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2$$

Since vectors are defined to transform like the displacement vector, it follows that
$$|\vec{A}|^2 \equiv \vec{A} \cdot \vec{A}$$
$$\underset{\mathcal{O}}{\doteq} -(A^0)^2 + (A^1)^2 + (A^2)^2 + (A^3)^2$$
is invariant as well.

Terminology: If $\vec{A} \cdot \vec{A} < 0$, $\vec{A}$ is "timelike"
$\vec{A} \cdot \vec{A} > 0$, $\vec{A}$ is "spacelike"
$\vec{A} \cdot \vec{A} = 0$, $\vec{A}$ is "null" or "light like"

More general notion of scalar product:
$$\vec{A} \cdot \vec{B} = -A^0 B^0 + A^1 B^1 + A^2 B^2 + A^3 B^3 \quad \text{in frame } \mathcal{O}$$

This is also clearly invariant. Proof: define $\vec{C} = \vec{A} + \vec{B}$. Then,
$$\vec{C} \cdot \vec{C} = \vec{A} \cdot \vec{A} + \vec{B} \cdot \vec{B} + 2 \vec{A} \cdot \vec{B}$$

invariant invariant

must be invariant.

Another way of writing this:

$$\vec{A}\cdot\vec{B} = (A^{\alpha}\vec{e}_{\alpha})\cdot(B^{\beta}\vec{e}_{\beta})$$

$$= A^{\alpha}B^{\beta}\,\vec{e}_{\alpha}\cdot\vec{e}_{\beta}$$

$$\equiv A^{\alpha}B^{\beta}\,\eta_{\alpha\beta}$$

→ Totally frame invariant form!

$\eta_{\alpha\beta} \equiv$ metric tensor

$$\doteq \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$