

Recap: Vector is a quartet of numbers that transforms between inertial reference frames like the components of the displacement vector:

$$\vec{A} = (A^0, A^1, A^2, A^3) \rightarrow \{A^\alpha\}$$

$$A^{\bar{\mu}} = \Lambda^{\bar{\mu}}{}_{\alpha} A^{\alpha}$$

Basis vectors: set of 4 defined such that

$$\vec{A} = A^{\alpha} \vec{e}_{\alpha}$$

Scalar product: $\vec{A} \cdot \vec{B} = -A^0 B^0 + A^1 B^1 + A^2 B^2 + A^3 B^3$

Frame-invariant way to write this:

$$\vec{A} \cdot \vec{B} = (A^{\alpha} \vec{e}_{\alpha}) \cdot (B^{\beta} \vec{e}_{\beta})$$

$$= A^{\alpha} B^{\beta} \vec{e}_{\alpha} \cdot \vec{e}_{\beta}$$

$$\equiv A^{\alpha} B^{\beta} \eta_{\alpha\beta}$$

$\eta_{\alpha\beta} \equiv \vec{e}_{\alpha} \cdot \vec{e}_{\beta} \rightarrow$ "metric tensor"

$$\stackrel{\text{D}}{=} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(Turns out to have this representation in ALL IRFs!)

①

The metric is the mathematical object which attaches a notion of distance to the manifold:

$$ds^2 = d\vec{x} \cdot d\vec{x} = \eta_{\alpha\beta} dx^\alpha dx^\beta$$

Notice that $d\vec{x} = dx^\alpha \vec{e}_\alpha$. This defines \vec{e}_α as a "coordinate basis vector."

For rectilinear coordinates, knowing this isn't too meaningful. However, the notion of "coordinate bases" versus other kinds of bases is quite important when we dig into curvilinear coordinates.

Example:
$$d\vec{x}_{\sim} = dr \vec{e}_{\hat{r}} + r d\theta \vec{e}_{\hat{\theta}} + r \sin\theta d\phi \vec{e}_{\hat{\phi}}$$
$$= dr \vec{e}_r + d\theta \vec{e}_\theta + d\phi \vec{e}_\phi$$

$\vec{e}_{r,\theta,\phi}$ are coordinate basis vectors ...

$\vec{e}_{\hat{r},\hat{\theta},\hat{\phi}}$ are orthonormal basis vectors.

In coordinate bases, $d\vec{x} = dx^\alpha \vec{e}_\alpha = dx^{\bar{\mu}} \vec{e}_{\bar{\mu}}$

Emphasizes that the transformation matrix is

$$\Lambda^{\bar{\mu}}{}_\alpha = \partial x^{\bar{\mu}} / \partial x^\alpha$$

Other physically interesting vectors:

4-velocity of observer:

$$\vec{u} = \frac{d\vec{x}}{d\tau}$$

$d\tau \equiv$ time interval as measured along that guy's trajectory
= "proper time"

$$\equiv (1, \gamma \underline{v})$$

In rest frame of that observer, $\vec{u} = (1, \underline{0}) \equiv$ timelike basis vector!

4-momentum: $\vec{p} = m\vec{u}$ "rest mass" of an object
 $\equiv (E, \underline{p})$

Important physics: $\vec{u} \cdot \vec{u} = -\gamma^2 + \gamma^2 |\underline{v}|^2 = -1$

$$\therefore \vec{p} \cdot \vec{p} = -m^2 \rightarrow E^2 - |\underline{p}|^2 = m^2$$

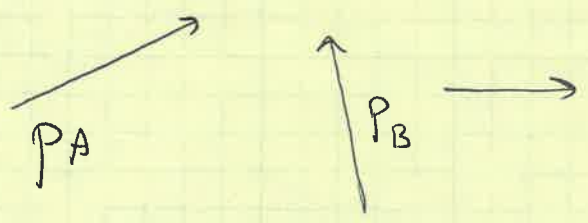
Conservation of energy & momentum: N particles interacting, then $\vec{p}_{TOT} = \sum_{i=1}^N \vec{p}_i$ is conserved.

Algebra simplifies if we choose "center of momentum" frame:

$$\vec{p}_{TOT}^{cm} \equiv (E, \underline{0})$$

Total energy of system.

Very useful for studying particle collisions:



m_c, m_D
establishes threshold to create new particles.

Invariance of dot product allows us to derive a very useful result:

Let \vec{p} be the 4-momentum of particle A.

Let \vec{u} be the 4-velocity of observer \mathcal{D} .

What does observer \mathcal{D} measure as A's energy?

\mathcal{D} will represent \vec{p} as

$$\vec{p} \stackrel{\mathcal{D}}{=} (\overset{\uparrow}{E}, \vec{p})$$

what we want. p^0 according to \mathcal{D} .

But, in \mathcal{D} 's rest frame, $\vec{u} = (1, \vec{0})$, so

$$E = -\vec{p} \cdot \vec{u}.$$

Invariance means this works in ANY frame!

Another important quantity: 4-acceleration.

$$\vec{a} = \frac{d\vec{u}}{d\tau} \quad \text{Simple to prove that } \vec{u} \cdot \vec{a} = 0:$$

$$\vec{u} \cdot \vec{u} = -1$$

$$\frac{d(\vec{u} \cdot \vec{u})}{d\tau} = 0 \rightarrow$$

$$\vec{u} \cdot \frac{d\vec{u}}{d\tau} = 0$$

Tensors: we've so far encountered one object that I explicitly called a "tensor" - the spacetime metric:

$$\eta_{\alpha\beta} = \vec{e}_\alpha \cdot \vec{e}_\beta$$

Tool for taking inner product of two vectors:

$$\vec{A} \cdot \vec{B} = A^\alpha B^\beta \eta_{\alpha\beta}$$

And, attaches notion of "distance" to manifold: $ds^2 = |d\vec{x}|^2 = dx^\alpha dx^\beta \eta_{\alpha\beta}$.

More generally: tensors are mathematical machines - functions - which are linear operators on vectors. Precise definition (Schutz):

A tensor of type $\binom{0}{N}$ is a function (or mapping) of N vectors into the 1 real numbers which is linear in each of the N arguments.
 LORENTZ INVARIANT

or dot product
Metric¹ is a $\binom{0}{2}$ tensor:

$$\vec{A} \cdot \vec{B} = a \quad \rightarrow \quad 2 \text{ vectors map to a real.}$$

$$\left. \begin{aligned} (\alpha \vec{A}) \cdot \vec{B} &= \alpha (\vec{A} \cdot \vec{B}) \\ (\vec{A} + \vec{C}) \cdot \vec{B} &= \vec{A} \cdot \vec{B} + \vec{C} \cdot \vec{B} \end{aligned} \right\} \text{linear on 1st "argument"}$$

$$\left. \begin{aligned} \vec{A} \cdot (\beta \vec{B}) &= \beta (\vec{A} \cdot \vec{B}) \\ \vec{A} \cdot (\vec{B} + \vec{C}) &= \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C} \end{aligned} \right\} \text{linear on 2nd arg.}$$

In very abstract form, we define the metric tensor $\vec{\eta}$ as a "2 slot" function that takes vectors as inputs and gives the dot product as output:

$$\vec{\eta}(\vec{A}, \vec{B}) = \vec{A} \cdot \vec{B} = \eta_{\alpha\beta} A^\alpha B^\beta$$

boldface
in text \rightarrow

Linearity tells us

$$\bar{\eta}(\alpha \vec{A} + \beta \vec{B}, \vec{C}) = \alpha \bar{\eta}(\vec{A}, \vec{C}) + \beta \bar{\eta}(\vec{B}, \vec{C})$$

etc.

Tensors take vectors - frame independent, geometric objects - as input, and gives a real number as output (likewise).

The tensor must therefore itself be a geometric object!

It will have different representations - different components - in different reference frames, but the real number we get out when we fill all its "slots" with vectors must be ~~the~~ invariant to all frames.

Components of a tensor: We get components out simply by putting unit vectors into the tensor's slots:

Low example: $\bar{\eta}(\vec{e}_\alpha, \vec{e}_\beta) = \vec{e}_\alpha \cdot \vec{e}_\beta = \eta_{\alpha\beta}$

Should be clear how to make a representation of a tensor in another frame: Suppose we have $\vec{e}_{\bar{\alpha}} = \Lambda^{\mu}_{\bar{\alpha}} \vec{e}_\mu$. Then,

$$\begin{aligned} \bar{\eta}(\vec{e}_{\bar{\alpha}}, \vec{e}_{\bar{\beta}}) &= \eta_{\bar{\alpha}\bar{\beta}} \\ &= \bar{\eta}(\Lambda^{\mu}_{\bar{\alpha}} \vec{e}_\mu, \Lambda^{\nu}_{\bar{\beta}} \vec{e}_\nu) \\ &= \Lambda^{\mu}_{\bar{\alpha}} \Lambda^{\nu}_{\bar{\beta}} \bar{\eta}(\vec{e}_\mu, \vec{e}_\nu) \end{aligned}$$

$$\rightarrow \eta_{\bar{\alpha}\bar{\beta}} = \Lambda^{\mu}_{\bar{\alpha}} \Lambda^{\nu}_{\bar{\beta}} \eta_{\mu\nu}$$

Note: "Line up the indices" is the rule.

Note 2: stupid example since $\bar{\eta} = \text{diag}(-1, 1, 1, 1)$ in all Lorentz frames.

$\binom{0}{1}$ tensors: special subset known as 1-forms
(sometimes referred to as "dual vectors", esp in older texts.)

A 1-form is a mapping from a single vector to real numbers:
 $\tilde{p}(\vec{A}) = \text{some real.}$

By inheriting tensor properties, easy to see that 1-forms satisfy axioms needed to define a vector space:

If \tilde{p} and \tilde{q} are both ~~both~~ 1-forms, then

$$\left. \begin{aligned} \tilde{s} &= \tilde{p} + \tilde{q} \\ \tilde{r} &= \alpha \tilde{p} \end{aligned} \right\} \text{ are also 1-forms}$$

$$\tilde{s}(\vec{A}) = \tilde{p}(\vec{A}) + \tilde{q}(\vec{A})$$

$$\tilde{r}(\vec{A}) = \alpha \tilde{p}(\vec{A})$$

Components: extract using basis vectors (tensor rule)

$$p_\alpha = \tilde{p}(\vec{e}_\alpha)$$

Gives us a very nice understanding of what the real number is when we apply 1-form to a vector:

$$\begin{aligned} \tilde{p}(\vec{A}) &= \tilde{p}(A^\alpha \vec{e}_\alpha) \\ &= A^\alpha \tilde{p}(\vec{e}_\alpha) \end{aligned}$$

$$= A^\alpha p_\alpha$$

This operation is called contraction.

Transformation of 1-form components:

$$\begin{aligned} p_{\tilde{\alpha}} &= \tilde{p}(\tilde{e}_{\tilde{\alpha}}) \\ &= \tilde{p}(\Lambda^{\mu}_{\tilde{\alpha}} \tilde{e}_{\mu}) = \Lambda^{\mu}_{\tilde{\alpha}} \tilde{p}(\tilde{e}_{\mu}) \\ &= \Lambda^{\mu}_{\tilde{\alpha}} p_{\mu} \rightarrow \text{line up indices!} \end{aligned}$$

Compare some transformation formulas:

Unit vectors: $\tilde{e}_{\tilde{\alpha}} = \Lambda^{\mu}_{\tilde{\alpha}} \tilde{e}_{\mu}$

1-form components: $p_{\tilde{\alpha}} = \Lambda^{\mu}_{\tilde{\alpha}} p_{\mu}$ ← "covaries" with basis - "covariant component"

Vector components: $A^{\tilde{\alpha}} = \Lambda^{\tilde{\alpha}}_{\mu} A^{\mu}$ ← "contravaries" w.r.t. basis - "contravariant comp."

Basis 1-forms: Want to define a set of 1-forms, $\{\tilde{\omega}^{\alpha}\}$ such that $\tilde{p} = p_{\alpha} \tilde{\omega}^{\alpha}$. (Note: α is not a component, but labels member of set.)

Originally defined components using basis vectors:

$$p_{\alpha} = \tilde{p}(\tilde{e}_{\alpha})$$

Now, define basis 1-forms using rule $\tilde{p}(\vec{A}) = p_{\alpha} A^{\alpha}$:

$$\begin{aligned} p_{\alpha} A^{\alpha} &\equiv \tilde{p}(\vec{A}) \\ &= p_{\beta} \tilde{\omega}^{\beta}(A^{\alpha} \tilde{e}_{\alpha}) \\ &= p_{\beta} A^{\alpha} \tilde{\omega}^{\beta}(\tilde{e}_{\alpha}) \end{aligned}$$

$$\rightarrow \boxed{\tilde{\omega}^{\beta}(\tilde{e}_{\alpha}) = \delta^{\beta}_{\alpha}}$$

Take this redefining relation for a basis 1-form. An example representation that works is

$$\tilde{\omega}^0 \doteq (1, 0, 0, 0)$$

$$\tilde{\omega}^1 \doteq (0, 1, 0, 0)$$

$$\tilde{\omega}^2 \doteq (0, 0, 1, 0)$$

$$\tilde{\omega}^3 \doteq (0, 0, 0, 1)$$

Looks like basis vectors, but not! Difference in this representation like row vs column vectors.

Last detail: transformations: $\tilde{\omega}^\mu = \Lambda^\mu_\nu \tilde{\omega}^\nu$. Easy to show.

Aside on dual nature:

$$\sum_{\mu=0}^3 p_\mu A^\mu \rightarrow \text{can do the calc, but doesn't have any real meaning.}$$

$$\sum_{\mu=0}^3 p_\mu A^\mu \rightarrow \text{meaningful!} \cdot \tilde{p}(\vec{A})$$

Akin to $\int \psi \phi d^3x \rightarrow \text{meaningless.}$

$$\int \psi^* \phi d^3x \rightarrow \text{meaningful} = \langle \psi | \phi \rangle.$$