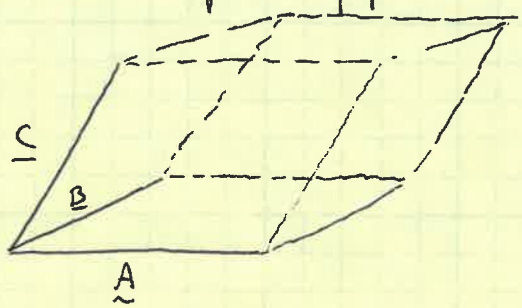
Volumes + volume integrals: Begin 3-D:

Consider a parallelepiped with sides $\underset{\sim}{A}$, $\underset{\sim}{B}$, and $\underset{\sim}{C}$:



$$\begin{aligned} \text{3-volume} &= \underset{\sim}{A} \cdot (\underset{\sim}{B} \times \underset{\sim}{C}) \\ &= \underset{\sim}{B} \cdot (\underset{\sim}{C} \times \underset{\sim}{A}) \\ &= \underset{\sim}{C} \cdot (\underset{\sim}{A} \times \underset{\sim}{B}) \end{aligned}$$

Nicely expressed as a determinant:

$$\text{3-vol} = \det \left\| \begin{matrix} A^1 & A^2 & A^3 \\ B^1 & B^2 & B^3 \\ C^1 & C^2 & C^3 \end{matrix} \right\| = \epsilon_{ijk} A^i B^j C^k$$

ϵ_{ijk} = Levi-Civita symbol

$\epsilon_{123} = +1 \rightarrow$ even permutations

$\epsilon_{132} = -1 \rightarrow$ odd permutations

$\epsilon_{112} = 0 \rightarrow$ any repeated index.

Levi-Civita symbol = components of a $\binom{0}{3}$ tensor.

$$\text{3 Volume} = V^3 = \bar{\bar{\epsilon}}(\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{C})$$

Notice, if we only put in 2 vectors, we get a 1-form:

$$\tilde{\Sigma} = \bar{\bar{\epsilon}}(-, \underset{\sim}{B}, \underset{\sim}{C})$$

or

$$\Sigma_i = \epsilon_{ijk} B^j C^k$$

This 1-form has the magnitude of the surface area corresponding to the face spanned by $\underset{\sim}{B}$ and $\underset{\sim}{C}$. 1-form → levels surface - makes sense!

With these notions, we can rigorously spell out Gauss's theorem in geometric language:

$$\int_{V^3} (\underset{\sim}{\nabla} \cdot \underset{\sim}{A})\, dV = \int_{\partial V^3} \underset{\sim}{A} \cdot d\underset{\sim}{\Sigma}$$

Need a differential triple: $d\vec{x}_1,\ d\vec{x}_2,\ d\vec{x}_3$

e.g.: $d\vec{x}_1 = dx\,\vec{e}_x\ ,\quad d\vec{x}_2 = dy\,\vec{e}_y\ ,\quad d\vec{x}_3 = dz\,\vec{e}_z$

Then: $dV = \epsilon_{ijk}\, dx_1^i\, dx_2^j\, dx_3^k = dx\,dy\,dz$

$d\Sigma_i = \epsilon_{ijk}\, dx_1^j\, dx_2^k$

$\hookrightarrow$ which legs you include depends on the particular face for which you do ∂V^3.

Spacetime generalization:

Consider "volume" enclosed by a parallelepiped with sides $\vec{A}, \vec{B}, \vec{C}, \vec{D}$: $\quad$ 4-vol $\equiv \epsilon_{\alpha\beta\gamma\delta} A^\alpha B^\beta C^\gamma D^\delta$ $\qquad \epsilon_{0123} = +1$, $\epsilon_{1023} = -1$ etc

The "area" of each "face" is actually a 3-volume, but oriented:

$$\Sigma_\alpha = \epsilon_{\alpha\beta\gamma\delta} B^\beta C^\gamma D^\delta$$

Gauss's Thm carries over to the generalization:

$$\int_{V^4} (\partial_\alpha V^\alpha)\, d^4x = \int_{\partial V^4} V^\alpha\, d\Sigma_\alpha$$

$$d^4x = \epsilon_{\alpha\beta\gamma\delta}\, dx_0^\alpha\, dx_1^\beta\, dx_2^\gamma\, dx_3^\delta$$

$$d\Sigma_\alpha = \epsilon_{\alpha\beta\gamma\delta}\, dx_1^\beta\, dx_2^\gamma\, dx_3^\delta$$

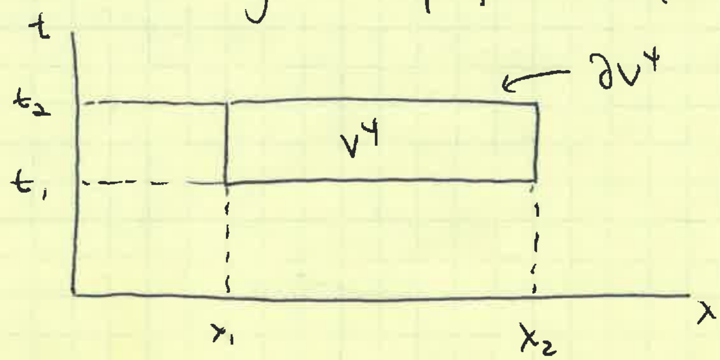
Apply this to our flux 4-vector for dust:

$$\int_{V^4} (\partial_\alpha N^\alpha)\, d^4x = \int_{\partial V^4} N^\alpha d\Sigma_\alpha$$

Conservation law: $\partial_\alpha N^\alpha = 0$

$$\rightarrow \quad \int_{\partial V^4} N^\alpha d\Sigma_\alpha = 0$$

Examine this for a specific volume / choice of inertial frame:



$$\int_{\partial V^4} N^\alpha d\Sigma_\alpha = \int_{t=t_2} N^0 dx\,dy\,dz - \int_{t=t_1} N^0 dx\,dy\,dz$$
$$+ \int_{x=x_2} N^1 dt\,dy\,dz - \int_{x=x_1} N^1 dt\,dy\,dz + \cdots$$

Now, consider $t_2 \rightarrow t_1 + dt$. Then, rearrange:

$$\int_{t_1+dt} N^0 dx\,dy\,dz - \int_{t_1} N^0 dx\,dy\,dz = -dt\left[\int_{x_2} N^1 dy\,dz - \int_{x_1} N^1 dy\,dz + \cdots\right]$$

$$\rightarrow \quad \frac{\partial}{\partial t}\int_{V^3} N^0 dx\,dy\,dz = -\int_{\partial V^3} \underset{\sim}{N} \cdot d\underset{\sim}{a}$$

Reduces to whe we wrote down intuitively earlier!

Emphasize: special case of general covariant form

$$\int_{\partial V^4} N^\alpha d\Sigma_\alpha = 0 .$$

More physically interesting example: Electric current

$$\vec{J} \doteq (\rho, \underset{\sim}{J}) \quad \ldots \text{ also satisfies a continuity equation:}$$

$$\partial_\alpha J^\alpha = 0$$

Much more interesting: connect to a field equation

$$\left.\begin{aligned} \underset{\sim}{\nabla} \cdot \underset{\sim}{E} &= 4\pi\rho \\ \underset{\sim}{\nabla} \times \underset{\sim}{B} - \frac{\partial \underset{\sim}{E}}{\partial t} &= 4\pi \underset{\sim}{J} \end{aligned}\right\} \text{ sourced}$$

$$\left.\begin{aligned} \underset{\sim}{\nabla} \times \underset{\sim}{E} + \frac{\partial \underset{\sim}{B}}{\partial t} &= 0 \\ \underset{\sim}{\nabla} \cdot \underset{\sim}{B} &= 0 \end{aligned}\right\} \text{ un sourced}$$

Written out in components:

$$\partial_i E_i = 4\pi\rho = 4\pi J^0$$

$$\epsilon_{ijk} \partial_j B_k - \partial_0 E_i = 4\pi J^i$$

$$\epsilon_{ijk} \partial_j E_k + \partial_0 B_i = 0$$

$$\partial_i B_i = 0$$

Note: position of spatial indices meaningless!

Define components of electromagnetic field tensor:

$$F^{0i} = E^i$$

$$F^{ij} = \epsilon^{ijk} B_k \quad \leftarrow \text{antisymmetric}$$

$$F^{i0} = -E^i \quad \leftarrow \text{so chosen thusly to maintain antisymmetry}$$

Representation:

$$F^{\mu\nu} \doteq \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{bmatrix}$$

Field equations become

$$\partial_\nu F^{\mu\nu} = 4\pi J^\mu$$

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0$$

Note: $$F_{\mu\nu} \doteq \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix}$$

key point:

Field equation <u>automatically</u> expresses conservation of current!

$$\begin{aligned} 4\pi\, \partial_\mu J^\mu &= \partial_\mu \partial_\nu F^{\mu\nu} \\ &= \partial_\nu \partial_\mu F^{\nu\mu} && \leftarrow \text{dummy indices} \\ &= -\partial_\nu \partial_\mu F^{\mu\nu} && \leftarrow \text{antisymmetry} \\ &= -\partial_\mu \partial_\nu F^{\mu\nu} && \leftarrow \text{symmetry} \\ &= 0 \end{aligned}$$

"Symmetry-antisymmetry trick" – worth knowing!

"Covariant" formulation of field equations - build conservation laws into the mathematical structure of the field equations.

Equation of motion:

$$m a^\mu = q F^{\mu\nu} u_\nu \qquad a^\mu = \frac{du^\mu}{d\tau}$$

or

$$\frac{du^\mu}{d\tau} = \frac{q}{m} F^{\mu\nu} u_\nu$$

Now contract with u_μ:

$$u_\mu \frac{du^\mu}{d\tau} = \frac{q}{m} F^{\mu\nu} u_\nu u_\mu$$

antisymmetric on exchange of indices

symmetric on exchange.

$$\rightarrow \quad u_\mu \frac{du^\mu}{d\tau} = 0$$

Already knew this!

$$u_\mu \frac{du^\mu}{d\tau} = \frac{1}{2} \frac{d}{d\tau} \left(u_\mu u^\mu \right)$$

$$= \frac{1}{2} \frac{d}{d\tau} (-1)$$

$$= 0.$$

Now consider energy & momentum of dust. Suppose each particle has a rest mass m.

In the rest frame, the <u>rest</u> <u>energy</u> <u>density</u> of the dust is

$$\rho = \rho_0 = m n_0$$

Now, go into the frame $\bar{O}$ that moves with $\underset{\sim}{v}$ relative to the "rest" frame: The number density goes up by γ, as does the energy / particle:

$$\rho = (\gamma m)(\gamma n_0)$$
$$= \gamma^2 \rho_0$$

This is <u>NOT</u> the transformation law of a 4-vector component! Not surprising: m is a component of $\vec{p}$, n_0 is a component of $\vec{N}$ - what we've assembled is the tensor product of two vectors. In particular, we've made the <u>STRESS-ENERGY</u> tensor:

$$\overline{\overline{T}} = \vec{N} \otimes \vec{p} = n_0 m \, \vec{u} \otimes \vec{u}$$
$$= \rho_0 \, \vec{u} \otimes \vec{u}$$

or

$$\boxed{T^{\alpha\beta} = \rho_0 u^\alpha u^\beta}$$

Physical meaning: $T^{\alpha\beta} = \bar{\bar{T}}(\tilde{d}x^\alpha, \tilde{d}x^\beta)$

= flux of α component of 4-momentum in the β direction

Consider meaning component by component:

T^{00} = flux of p^t in t direction
= energy density

T^{0i} = flux of p^0 in x^i direction
= energy flux in x^i

T^{i0} = flux of p^i in t direction
= i - momentum density

T^{ij} = flux of p^i in x^j direction

Example in some frame: $\vec{u} = (\gamma, \gamma \underset{\sim}{v})$

$$\left.\begin{aligned} T^{00} &= \gamma^2 \rho_0 \\ T^{0i} &= \gamma^2 \rho_0 v^i \\ T^{i0} &= \gamma^2 \rho_0 v^i \\ T^{ij} &= \gamma^2 \rho_0 v^i v^j \end{aligned}\right\} \text{Note symmetry!}$$

Most of the universe is not dust! Could be a fluid with temperature, pressure, viscosity, ... or whatever.

Deduce how to construct $T^{\alpha\beta}$ by considering example components in some frame. I.e., no set recipe, just follow the physics.

T^{00} = energy density = ρ

$\neq \rho_0$ in general! Only for dust in rest frame and zero temp fluids in rest frame.

T^{0i} = energy flux. Not uncommon to be zero in many IRFs. Arises from bulk motion, heat transport, etc.

T^{i0} = momentum density

Note $T^{i0} = T^{0i}$

T^{0i} = energy density times flow velocity.

T^{i0} = mass density times flow velocity

T^{ij} = stress

T^{ii} = pressure

T^{ij}, $i \neq j$ "rigidity" or transport force - like viscosity.

Examples: Perfect fluid. No energy flow in "rest" frame

$\rightarrow$ no heat transport

No lateral stresses (not wet...)

$\rightarrow$ no viscosity

$\rightarrow$ Characterized totally by ρ and pressure P!

In a given frame,

$$T^{\alpha\beta} \doteq \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{bmatrix}$$

Clearly get ρ from $\rho \vec{u} \otimes \vec{u}$. To get the pressure pieces, use projection tensor developed in part 1.

Total: $$\overline{\overline{T}} = \rho \vec{u} \otimes \vec{u} + P(\overline{\overline{\eta}} + \vec{u} \otimes \vec{u})$$

or $$T_{\alpha\beta} = \rho u_\alpha u_\beta + P(\eta_{\alpha\beta} + u_\alpha u_\beta)$$

$$= (\rho + P) u_\alpha u_\beta + P \eta_{\alpha\beta}$$

Final point: Newtonian field equation is

$$\nabla^2 \phi = 4\pi G \rho$$

or $$\partial^i \partial_i \phi = 4\pi G \rho$$

We can't have just ρ on RHS in relativity – it's a component of a tensor. To have an equation with geometric meaning, need the whole tensor.

RHS $\rightarrow T_{\mu\nu}$.

Much of the next few weeks devoted to figuring out the LHS. Potential will go over to a 2-index geometric object – the spacetime metric.