Recap: Introduced stress-energy tensor

$T^{\alpha\beta} \equiv$ flux of p^α in x^β direction

T^{00} = energy density in some frame

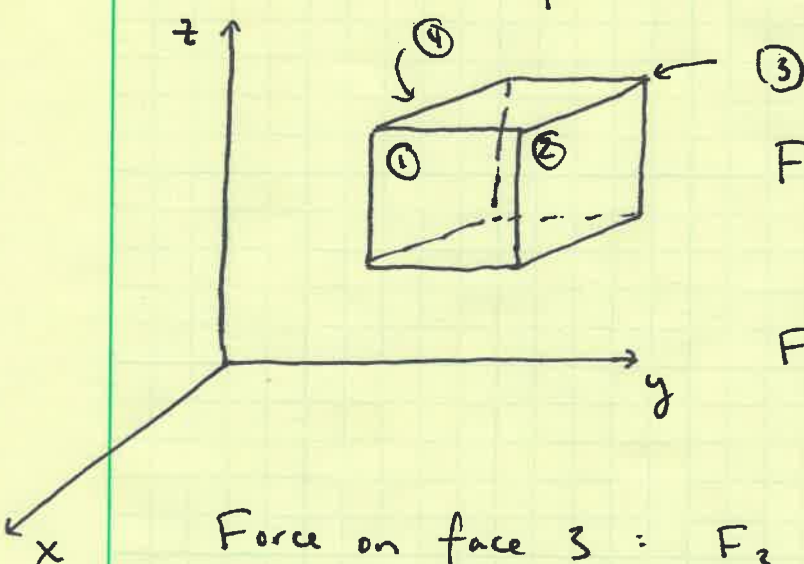
T^{0j} = energy flux in x^j direction

T^{i0} = density of momentum p^i

T^{ij} = stress

Tensor is symmetric: $T^{\alpha\beta} = T^{\beta\alpha}$. If $T^{ij} \neq T^{ji}$, physically absurd situation could develop.

Consider cube of sides ℓ embedded in $T^{\alpha\beta}$:



Force on face 1:

$$\underset{\sim}{F}_1 \doteq \{ T^{ix} \ell^2 \}$$

Force on face 2:

$$\underset{\sim}{F}_2 \doteq \{ T^{iy} \ell^2 \}$$

Force on face 3: $\underset{\sim}{F}_3 \simeq -\underset{\sim}{F}_1$ ← exact as $\ell \to 0$

Force on face 4: $\underset{\sim}{F}_4 \simeq -\underset{\sim}{F}_2$

Torques due to these forces:

$$\underset{\sim}{T}_1 \rightarrow \tau_1^{z} = -xF_1^{y} = -x\,T^{yx}\ell^2$$
$$= -\frac{1}{2}T^{yx}\ell^3$$

$$\underset{\sim}{T}_3 \rightarrow \tau_3^{z} = \tau_1^{z}$$

$$\underset{\sim}{T}_2 \rightarrow \tau_2^{z} = y F_2^{x} = y\,T^{xy}\ell^2$$
$$= \frac{1}{2}T^{xy}\ell^3$$

$$\underset{\sim}{T}_4 \rightarrow \tau_4^{z} = \tau_2^{z}$$

Add to find total: $\tau^z = \ell^3\left(T^{xy} - T^{yx}\right)$

Moment of inertia of cube: $I = \alpha(\rho\ell^3)\ell^2$

Coefficient of order unity

Angular acceleration of the cube:

$$\ddot{\theta} = \tau^z / I$$
$$\propto \left(T^{xy} - T^{yx}\right)/\ell^2$$

In order for cube to not start spontaneously rotating, we must have $T^{xy} = T^{yx}$ ...

Considering other sides yields $T^{ij} = T^{ji}$.

Conservation of energy and momentum:

$$\boxed{\partial_\alpha T^{\alpha\beta} = 0}$$

Pick a particular frame, can break conservation principle into c of energy and c of m. In general, only a combined law makes sense.

$$\partial_\alpha T^{0\alpha} = 0 \qquad \text{conservation of energy}$$

$$\frac{\partial \rho}{\partial t} = - \frac{\partial T^{0i}}{\partial x^i}$$

rate of change of energy density

divergence of energy flux

Can be converted into an integral equation:

$$\frac{\partial}{\partial t} \int_{V^3} \rho \, d^3x = - \int_{\partial V^3} T^{0i} \, d\Sigma_i$$

$dE/dA\,dt \rightarrow$ luminosity

$$\partial_\alpha T^{i\alpha} = 0 \rightarrow \text{conservation of momentum}$$

$$\rightarrow \frac{\partial T^{i0}}{\partial t} = - \frac{\partial T^{ij}}{\partial x^j} \qquad (\text{or } \partial_t T^{i0} = - \partial_j T^{ij})$$

$$\partial_t \int_{V^3} T^{i0} \, d^3x = - \int_{\partial V^3} T^{ij} \, d\Sigma_j$$

Important example: "Perfect" fluid

No energy flow in rest frame

$\rightarrow$ no heat transport

No lateral stress

$\rightarrow$ no viscosity. (Dry fluid)

Fluid is totally characterized by density ρ and pressure P:

There exists a frame in which

$$T^{\alpha\beta} \doteq \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{bmatrix}$$

(Notion of equation of state: relate P & ρ, $P = P(\rho)$.)

Want to make the geometric construction of this. The ρ piece clearly comes from $\rho\vec{u}\otimes\vec{u}$, where $\vec{u}$ is 4-vel of the "rest" frame. The P bits come from the projection tensor developed on part 1:

$$\overline{\overline{T}} = \rho\,\vec{u}\otimes\vec{u} + P(\overline{\overline{\eta}} + \vec{u}\otimes\vec{u})$$

or

$$T_{\alpha\beta} = \rho u_\alpha u_\beta + P(\eta_{\alpha\beta} + u_\alpha u_\beta)$$

$$= P\eta_{\alpha\beta} + (\rho + P)u_\alpha u_\beta$$

Another example: Point particle of rest mass m_0 moving on worldline $\vec{z}(\tau)$:

$$T_{\mu\nu} = m_0 \int d\tau \, u_\mu u_\nu \, \delta^4[\vec{x} - \vec{z}(\tau)]$$

$$\vec{u} = d\vec{z}/d\tau$$

Using $\delta^4[\vec{x} - \vec{z}(\tau)] = \delta[t - z^0(\tau)]\,\delta[x - z^1(\tau)] \cdots$

plus the rule

$$\int f(x)\,\delta[g(x)]\,dx = \frac{f(x_0)}{|g'|_{x=x_0}}$$

where x_0 is defined such that $g(x_0) = 0$.

we can integrate this up to

$$T_{\mu\nu} = \frac{m_0 u_\mu u_\nu}{u^0}\,\delta^3[\underset{\sim}{x} - \underset{\sim}{z}(\tau)]$$

Another: $T^{\mu\nu}_{E\&M} = \frac{1}{4\pi}\left[F^{\mu\lambda}F^{\nu}{}_{\lambda} - \frac{1}{4}\eta^{\mu\nu}F^{\lambda\sigma}F_{\lambda\sigma}\right]$

$$\rightarrow \quad T^{00} = \frac{\underset{\sim}{E}\cdot\underset{\sim}{E} + \underline{B}\cdot\underline{B}}{8\pi}$$

$$T^{0i} = \frac{(\underset{\sim}{E}\times\underline{B})^i}{4\pi}$$

$$T^{ij} = \frac{1}{8\pi}\left[(\underset{\sim}{E}\cdot\underset{\sim}{E} + \underline{B}\cdot\underline{B})\delta^{ij} - 2(E^iE^j + B^iB^j)\right]$$

Example: $\underset{\sim}{E} = E^x \underset{\sim}{e}_x$

$$T^{\mu\nu}_{EM} \doteq \frac{(E^x)^2}{8\pi}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Attractive "tension"

pressure

Follows from choosing Lagrangian density:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\mu J^\mu \qquad \left[L = \int d^3x\,\mathcal{L}\right]$$

Then build an action: $S = \int dt\, L = \int d^4x\,\mathcal{L}$

$T_{\mu\nu}$ pops out by varying metric!

$$T_{\mu\nu} = \frac{-2}{\sqrt{\det(\eta_{\mu\nu})}}\frac{\delta S}{\delta g^{\mu\nu}}$$

Prelude to curvature: flat spacetime, but curvilinear coordinates.

$$(t, r, \phi, z) \qquad x = r\cos\phi \qquad y = r\sin\phi$$

Continue to use a <u>coordinate basis</u>:

$$d\vec{x} = dx^\alpha \vec{e}_\alpha$$
$$= dt\,\vec{e}_t + dr\,\vec{e}_r + \underset{\text{angle}}{d\phi}\,\underset{\text{length}}{\vec{e}_\phi} + dz\,\vec{e}_z$$

Not a normal basis! $|\vec{e}_\phi \cdot \vec{e}_\phi| \neq 1$.

Transformation matrix: $L^\alpha{}_{\bar{\mu}} \equiv \dfrac{\partial x^\alpha}{\partial x^{\bar{\mu}}}$ (Reserve $\Lambda^\alpha{}_{\bar{\mu}}$ for Lorentz transformation.)

Let barred be polar, unbarred be cartesian:

$$\frac{\partial x^\alpha}{\partial x^{\bar{\mu}}} \doteq \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & \sin\phi & 0 \\ 0 & -r\sin\phi & r\cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = L^\alpha{}_{\bar{\mu}}$$

$$\frac{\partial x^{\bar{\mu}}}{\partial x^\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi/r & 0 \\ 0 & \sin\phi & \cos\phi/r & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = L^{\bar{\mu}}{}_\alpha$$

Basis vectors:

$$\vec{e}_r = \cos\phi\, \vec{e}_x + \sin\phi\, \vec{e}_y = L^\alpha{}_r\, \vec{e}_\alpha$$

$$\vec{e}_\phi = -r\sin\phi\, \vec{e}_x + r\cos\phi\, \vec{e}_y = L^\alpha{}_\phi\, \vec{e}_\alpha$$

Grows because the angle spans constant angle.

Metric: $g_{\alpha\beta} = \vec{e}_\alpha \cdot \vec{e}_\beta$

$$= \mathrm{diag}(-1, 1, r^2, 1)$$

Reserve $\eta_{\alpha\beta}$ for $\mathrm{diag}(-1, 1, 1, 1)$

Line element: $ds^2 = -dt^2 + dr^2 + r^2 d\phi^2 + dz^2$

Basis 1-forms: $\tilde{d}r = L^r{}_\alpha\, \tilde{d}x^\alpha = \cos\phi\, \tilde{d}x + \sin\phi\, \tilde{d}y$

$$\tilde{d}\phi = -\frac{\sin\phi}{r}\, \tilde{d}x + \frac{\cos\phi}{r}\, \tilde{d}y$$

Key place where this matters: Calculating derivatives. Need to include variation in basis vectors!

$$\frac{\partial \vec{e}_r}{\partial r} = 0 \qquad \frac{\partial \vec{e}_r}{\partial \phi} = \frac{\vec{e}_\phi}{r}$$

$$\frac{\partial \vec{e}_\phi}{\partial r} = \frac{\vec{e}_\phi}{r} \qquad \frac{\partial \vec{e}_\phi}{\partial \phi} = -r\vec{e}_r$$

$$\vec{V} = V^\alpha \vec{e}_\alpha \qquad \nabla \vec{V} = \partial_\beta \left(V^\alpha \vec{e}_\alpha\right) \tilde{\omega}^\beta$$

Components of tensor:

$$\frac{\partial \vec{V}}{\partial x^\beta} = \left(\frac{\partial V^\alpha}{\partial x^\beta}\right)\vec{e}_\alpha + V^\alpha \frac{\partial \vec{e}_\alpha}{\partial x^\beta}$$

$\partial \vec{e}/\partial x^\beta$ is something that we can write as linear combination of basis vectors:

$$\longrightarrow \quad \partial_\beta \vec{e}_\alpha = \frac{\partial \vec{e}_\alpha}{\partial x^\beta} = \Gamma^\mu{}_{\beta\alpha}\, \vec{e}_\mu$$

$\hookrightarrow$ "Christoffel Symbol"

<u>CAUTION</u>: Schutz, MTW define this as $\Gamma^\mu{}_{\alpha\beta}$.

$\longrightarrow$ Carroll, Wald define as here. Turns out to be irrelevant - symmetric for coordinates bases. (Will prove later.)

For polar coords, $\Gamma^\phi{}_{r\phi} = \Gamma^\phi{}_{\phi r} = 1/r$

$$\Gamma^r{}_{\phi\phi} = -r$$

All others - zero.

Return to derivative of vector:

$$\partial_\beta \vec{V} = \left(\partial_\beta V^\alpha \vec{e}_\alpha + V^\alpha \Gamma^\mu_{\beta\alpha} \vec{e}_\mu \right)$$

$$= \left(\partial_\beta V^\alpha \vec{e}_\alpha + V^\mu \Gamma^\alpha_{\beta\mu} \vec{e}_\alpha \right)$$

$$\rightarrow \quad \partial_\beta \vec{V} = \underbrace{\left(\partial_\beta V^\alpha + V^\mu \Gamma^\alpha_{\beta\mu} \right)}_{\text{The Covariant derivative}} \vec{e}_\alpha$$

$$\equiv \nabla_\beta V^\alpha \vec{e}_\alpha$$

$$\nabla_\beta V^\alpha = \partial_\beta V^\alpha + V^\mu \Gamma^\alpha_{\beta\mu}$$

Not individually components of a tensor! Only sum is tensorial!

Free indices placed to agree w/LHS; dummy last.

Components of tensor:

$$\nabla \vec{V} = \nabla_\beta V^\alpha \, \tilde{\omega}^\beta \otimes \vec{e}_\alpha$$

Example application: divergence.

$$\nabla_\alpha V^\alpha = \partial_\alpha V^\alpha + \Gamma^\alpha_{\alpha\mu} V^\mu$$

$$= \partial_t V^t + \partial_z V^z + \partial_r V^r + \partial_\phi V^\phi + \Gamma^\phi_{\phi r} V^r$$

$$= \partial_t V^t + \partial_z V^z + \partial_r V^r + \partial_\phi V^\phi + \frac{V^r}{r}.$$

Notice the term $\partial_\phi V^\phi$ – Is this dimensionally correct?

Consider the vector:

$$\vec{V} = V^\alpha \vec{e}_\alpha = V^t \vec{e}_t + V^r \vec{e}_r + V^\phi \vec{e}_\phi + V^z \vec{e}_z$$

Since $[\vec{e}_\phi]$ is length,

$[V^\phi] = [\vec{V}] / \text{length}$. So $\partial_\phi V^\phi$ is cool!

Derivs of other tensorial objects.

Scalars: No basis needed! So, no worries w complications:

$$\nabla_\alpha \phi = \partial_\alpha \phi$$

1-form: deduce by considering deriv of $p_\alpha A^\alpha \rightarrow$ scalar:

$$\nabla_\beta (p_\alpha A^\alpha) = \partial_\beta (p_\alpha A^\alpha)$$

$$= A^\alpha \partial_\beta p_\alpha + p_\alpha \partial_\beta A^\alpha$$

Now write $\partial_\beta A^\alpha = \nabla_\beta A^\alpha - \Gamma^\alpha_{\beta\mu} A^\mu$:

$$\nabla_\beta (p_\alpha A^\alpha) = A^\alpha \partial_\beta p_\alpha + p_\alpha \nabla_\beta A^\alpha - p_\alpha \Gamma^\alpha_{\beta\mu} A^\mu$$

$$= p_\alpha \nabla_\beta A^\alpha + A^\alpha (\partial_\beta p_\alpha - p_\mu \Gamma^\mu_{\beta\alpha})$$

Now, require covariant deriv to obey Leibnitz rule:

$$\nabla_\beta (p_\alpha A^\alpha) = p_\alpha \nabla_\beta A^\alpha + A^\alpha \nabla_\beta p_\alpha$$

Compare:

$$\boxed{\nabla_\beta p_\alpha = \partial_\beta p_\alpha - \Gamma^\mu_{\beta\alpha} p_\mu}$$

From this, simple to find rule for 1-form derivatives:

$$\partial_\beta \tilde{\omega}^\alpha = -\Gamma^\alpha_{\beta\mu}\tilde{\omega}^\mu$$

Minus sign enforces $\langle \tilde{\omega}^\alpha, \vec{e}_\beta \rangle = \delta^\alpha{}_\beta$. → (Can start with this, prove Leibnitz afterwards.)

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generalize further:

$$\nabla_\beta T^{\mu\nu} = \partial_\beta T^{\mu\nu} + \Gamma^\mu_{\beta\alpha} T^{\alpha\nu} + \Gamma^\nu_{\beta\alpha} T^{\mu\alpha}$$

$$\nabla_\beta T_{\mu\nu} = \partial_\beta T_{\mu\nu} - \Gamma^\alpha_{\beta\mu} T_{\alpha\nu} - \Gamma^\alpha_{\beta\nu} T_{\mu\alpha}$$

$$\nabla_\beta T^{\mu\nu\cdots}{}_{\rho\tau\cdots} = \partial_\beta T^{\mu\nu\cdots}{}_{\rho\tau\cdots} + \Gamma^\mu_{\beta\alpha} T^{\alpha\nu\cdots}{}_{\rho\tau\cdots} + \Gamma^\nu_{\beta\alpha} T^{\mu\alpha\cdots}{}_{\rho\tau\cdots} - \Gamma^\alpha_{\beta\rho} T^{\mu\nu\cdots}{}_{\alpha\tau\cdots} - \Gamma^\alpha_{\beta\tau} T^{\mu\nu\cdots}{}_{\rho\alpha\cdots}$$

Just go through and "correct" each index.

Definition $\partial_\beta \vec{e}_\alpha = \Gamma^\mu_{\beta\alpha} \vec{e}_\mu$ suggests we generate Christoffels by studying derivatives of basis vectors. Yuck.

Better way to get it: from the metric. Derivation will introduce a key property of tensor relationships:

A tensorial equation that holds in one representation must hold in <u>ALL</u> representations. CHANGING COORDINATES CANNOT CHANGE THE EQUATION!

Example application: "double gradient" of scalar.

Cartesian: $\nabla\nabla\phi \equiv \partial_\alpha \partial_\beta \phi \, \tilde{\omega}^\alpha \otimes \tilde{\omega}^\beta$

↳ Clearly symmetric!

general coordinates: $\nabla\nabla\phi = \nabla_\alpha \nabla_\beta \phi \, \tilde{\omega}^\alpha \otimes \tilde{\omega}^\beta$

Must be generally symmetric: $\nabla_\alpha \nabla_\beta \phi = \nabla_\beta \nabla_\alpha \phi$

$$\rightarrow \cancel{\partial_\alpha \partial_\beta \phi} - \Gamma^\mu_{\alpha\beta} \partial_\mu \phi = \cancel{\partial_\beta \partial_\alpha \phi} - \Gamma^\mu_{\beta\alpha} \partial_\mu \phi$$

$$\rightarrow \left(\Gamma^\mu_{\alpha\beta} - \Gamma^\mu_{\beta\alpha} \right) \partial_\mu \phi = 0$$

↳ Symmetry on Christoffels!

Important application: Gradient of metric.

$$\nabla \bar{\bar{g}} = \partial_\gamma \eta_{\alpha\beta}\, \tilde{\omega}^\alpha \otimes \tilde{\omega}^\beta \otimes \tilde{\omega}^\gamma = 0 \qquad \text{(Cartesian)}$$

$$= \nabla_\gamma g_{\alpha\beta}\, \tilde{\omega}^\alpha \otimes \tilde{\omega}^\beta \otimes \tilde{\omega}^\gamma \qquad \text{in general.}$$

We <u>require</u> $\nabla_\gamma g_{\alpha\beta} = 0$:

$$\text{I} \qquad \nabla_\gamma g_{\alpha\beta} = \partial_\gamma g_{\alpha\beta} - \Gamma^\mu_{\gamma\alpha} g_{\mu\beta} - \Gamma^\mu_{\gamma\beta} g_{\alpha\mu} = 0$$

$$\text{II} \qquad \nabla_\alpha g_{\beta\gamma} = \partial_\alpha g_{\beta\gamma} - \Gamma^\mu_{\alpha\beta} g_{\mu\gamma} - \Gamma^\mu_{\alpha\gamma} g_{\beta\mu} = 0$$

$$\text{III} \qquad \nabla_\beta g_{\gamma\alpha} = \partial_\beta g_{\gamma\alpha} - \Gamma^\mu_{\beta\gamma} g_{\mu\alpha} - \Gamma^\mu_{\beta\alpha} g_{\gamma\mu} = 0$$

Construct I − II − III:

$$\partial_\gamma g_{\alpha\beta} - \partial_\alpha g_{\beta\gamma} - \partial_\beta g_{\gamma\alpha}$$
$$- g_{\mu\beta}\cancelto{0}{\left(\Gamma^\mu_{\gamma\alpha} - \Gamma^\mu_{\alpha\gamma}\right)}$$
$$+ g_{\mu\gamma}\left(\Gamma^\mu_{\alpha\beta} + \Gamma^\mu_{\beta\alpha}\right)$$
$$+ g_{\mu\alpha}\cancelto{0}{\left(\Gamma^\mu_{\beta\gamma} - \Gamma^\mu_{\gamma\beta}\right)} = 0$$

$$\rightarrow \quad g_{\mu\gamma}\Gamma^\mu_{\alpha\beta} = \frac{1}{2}\left(\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\gamma\alpha} - \partial_\gamma g_{\alpha\beta}\right)$$

$$\equiv \Gamma_{\gamma\alpha\beta}$$

$$\Gamma^\mu{}_{\alpha\beta} = g^{\mu\gamma}\,\Gamma_{\gamma\alpha\beta}$$