Principle of equivalence

Key concept in special relativity was the notion of an <u>inertial</u> reference frame: A frame in which particles are unaccelerated if no forces act on them.

With gravity in the picture, we appear to lose this: stuff accelerates due to a "gravitational force".

However, gravity appears to be special: the "gravitational force" exerted on an object is proportional to its mass. In other words, "gravitational charge" is mass! This means all objects experience the same acceleration due to the field. In the absence of non-grav. forces, objects maintain their <u>relative</u> velocities.

This means there is <u>A</u> reference frame that ~~ca~~ captures the essence of an inertial reference frame: A frame that is freely falling! If we compute everything <u>relative</u> to that frame's acceleration, our intuitive notion of what gravity does is cancelled out.

We have formulated the "Weak Equivalence Principle":

~~Uniform gravitational fields cannot be distinguished from uniform acceleration.~~

The motion of freely falling particles ~~in a~~ due to gravity cannot be distinguished from uniform acceleration. (Tested: accurate to $\sim 10^{-13}$ level.)

Over sufficiently small regions,

So, can we "do gravity" by just using special relativity, but redefining the notion of an inertial frame? No:

1. The freely falling frames (FFF) are not the same at all points: we cannot cover all of ~~special relativity~~ spacetime with such a frame. Hence, global Lorentz symmetry is lost.

2. gravitational fields are not uniform: In a freely falling elevator, Accelerations are slightly different at top & bottom: Tides exist.

These tidal differences cannot be transformed away by jumping into a new frame. We will come to regard these tides as the real fundamental "gravitational field".

Note that tides cause initially "parallel" (in a spacetime sense) trajectories to become non-parallel: Our manifold violates Euclid's parallelism axiom! We have curvature.

Best we can do is define a frame that is "locally" Lorentzian, with size set by tides/curvature. In that frame, we can formulate a generalization of the WEP:

> In sufficiently small regions of spacetime, we can find a representation such that the laws of physics reduce to those of special relativity.

"Einstein Equivalence Principle"

(Strong Equivalence Principle: "gravitational energy" falls just like regular mass energy. Just FYI.)

How do we know we can always find a LOCALLY Lorentz frame (with limits on "local" set by "curvature")?

Let $\{x^\alpha\}$ be our starting coordinate system, metric is $g_{\alpha\beta}$.
Let $\{x^{\bar\alpha}\}$ be coordinates in which spacetime has a Lorentz representation in vicinity of some event $\mathcal{P}$:

$$\mathcal{P} \doteq \{x^\mu_p\} \doteq \{x^{\bar\mu}_p\}$$

Assume there is some mapping between coordinates: $x^\alpha = x^\alpha(x^{\bar\alpha})$
With transformation matrix $L^\alpha{}_{\bar\alpha} = \partial x^\alpha / \partial x^{\bar\alpha}$.

Goal: Find a coordinate transformation that lets us set

$$g_{\bar\mu\bar\nu} = L^\alpha{}_{\bar\mu} L^\beta{}_{\bar\nu}\, g_{\alpha\beta} = \eta_{\bar\mu\bar\nu}$$

over as large ~~an~~ ~~area~~ a region as possible.

What we'll do: Expand $(L^\alpha_{\bar\mu} L^\beta_{\bar\nu} g_{\alpha\beta})$ in a Taylor series about $\mathcal{P}$ and attack.

Key piece of analysis: Compare the number of degrees of freedom offered by the coordinate transformation (which we can select) to the constraints imposed by the metric (and its derivatives) which we're given.

Do expansion in barred (Lorentz frame) coordinates:

$$g_{\alpha\beta} = g_{\alpha\beta}|_{\wp} + (x^{\bar{\gamma}} - x^{\bar{\gamma}}_{p})\,\partial_{\bar{\gamma}}\, g_{\alpha\beta}|_{\wp}$$
$$+ \frac{1}{2}(x^{\bar{\gamma}} - x^{\bar{\gamma}}_{p})(x^{\bar{\delta}} - x^{\bar{\delta}}_{p})\,\partial_{\bar{\gamma}}\partial_{\bar{\delta}}\, g_{\alpha\beta}|_{\wp} + \cdots$$

$$L^{\alpha}{}_{\bar{\mu}} = L^{\alpha}{}_{\bar{\mu}}|_{\wp} + (x^{\bar{\gamma}} - x^{\bar{\gamma}}_{p})\,\partial_{\bar{\gamma}}\, L^{\alpha}{}_{\bar{\mu}}|_{\wp}$$
$$+ \frac{1}{2}(x^{\bar{\gamma}} - x^{\bar{\gamma}}_{p})(x^{\bar{\delta}} - x^{\bar{\delta}}_{p})\,\partial_{\bar{\gamma}}\partial_{\bar{\delta}}\, L^{\alpha}{}_{\bar{\mu}}|_{\wp}$$

$g_{\alpha\beta}|_{p}$, $\partial g_{\alpha\beta}$, $\partial^2 g_{\alpha\beta}$: given to us, provide <u>constraints</u>.

$L^{\alpha}{}_{\bar{\mu}}$, $\partial L^{\alpha}{}_{\bar{\mu}}$, $\partial^2 L^{\alpha}{}_{\bar{\mu}}$: freely specifiable, provide degrees of freedom.

Logic of calculation: make

$$L^{\alpha}{}_{\bar{\mu}} L^{\beta}{}_{\bar{\nu}}\, g_{\alpha\beta} = (L^{\alpha}{}_{\bar{\mu}}|_{p})(L^{\beta}{}_{\bar{\nu}}|_{\wp})\, g_{\alpha\beta}|_{\wp} \quad ⓪$$
$$+ (x^{\bar{\gamma}} - x^{\bar{\gamma}}_{p})\left(\text{terms involving } \partial_{\bar{\gamma}} g_{\alpha\beta}|_{\wp},\ \partial_{\bar{\gamma}} L^{\alpha}{}_{\bar{\mu}}|_{\wp}\right) \quad ①$$
$$+ \frac{1}{2}(x^{\bar{\gamma}} - x^{\bar{\gamma}}_{p})(x^{\bar{\delta}} - x^{\bar{\delta}}_{p})\left[\text{terms involving 2ND derivs}\right] \quad ②$$
$$+ \cdots$$

as Lorentzian as possible.

0^{TH} order: $g_{\alpha\beta}|_{\mathcal{P}}$: symmetric 4×4

$\rightarrow$ 10 conditions

$L^{\alpha}{}_{\bar{\mu}}|_{\mathcal{P}} \equiv \left.\dfrac{\partial x^{\alpha}}{\partial x^{\bar{\mu}}}\right|_{\mathcal{P}}$: not symmetric 4×4

$\rightarrow$ 16 Degrees of freedom.

Freedom to choose $L^{\alpha}{}_{\bar{\mu}}|_{\mathcal{P}}$ to set $g_{\bar{\mu}\bar{\nu}} \equiv \eta_{\bar{\mu}\bar{\nu}}$ at $\mathcal{P}$ with 6 D.O.F. left over! 3 boosts, 3 rotations.

$\partial_{\bar{\gamma}} g_{\alpha\beta}|_{\mathcal{P}}$ = (symmetric 4×4 on $\alpha\beta$) $\times$ (4 components $\bar{\gamma}$)

$\rightarrow$ 40 conditions we must satisfy.

$\left(\partial_{\bar{\gamma}} L^{\alpha}{}_{\bar{\mu}}\right)|_{\mathcal{P}} = \dfrac{\partial^2 x^{\alpha}}{\partial x^{\bar{\gamma}} \partial x^{\bar{\mu}}}$ = (4 components α) $\times$ (symmetric 4×4 on $\bar{\gamma}$ and $\bar{\mu}$)

$\rightarrow$ 40 D.O.F. to meet conditions. MATCH!

$\partial_{\bar{\gamma}} \partial_{\bar{\delta}} g_{\alpha\beta}|_{\mathcal{P}}$ = (symmetric 4×4 on $\bar{\gamma}\bar{\delta}$) $\times$ (symmetric 4×4 on $\alpha\beta$)

$\rightarrow$ 100 conditions we must satisfy

$\partial_{\bar{\gamma}} \partial_{\bar{\delta}} L^{\alpha}{}_{\bar{\mu}}|_{\mathcal{P}} = \dfrac{\partial^3 x^{\alpha}}{\partial x^{\bar{\mu}} \partial x^{\bar{\delta}} \partial x^{\bar{\gamma}}}$ = (symmetric $4\times 4\times 4$ on $\bar{\mu}\bar{\gamma}\bar{\delta}$) $\times$ (4 components α)

$= 4 \times \left.\dfrac{n(n+1)(n+2)}{3!}\right|_{n=4}$

$= 80$ D.O.F.

NOT ENOUGH!

We <u>cannot</u> transform away the 2^{ND} deriv terms.

Coordinate freedom lets us put the metric in the form

$$g_{\bar\mu\bar\nu} = \eta_{\bar\mu\bar\nu} + \mathcal{O}\Big[\underbrace{(\partial^2 g)}_{\text{2ND derivs of metric}}\underbrace{(\delta x)^2}_{\text{coordinate distance from event } \mathcal{P}}\Big]$$

Key things to note:

1. Frame is Lorentz (or "inertial") only up to corrections that scale like the 2ND deriv of the metric. Keys into intuition about tides: Tides look like 2ND derivative of potential. Suggests "metric" will play a role in our theory analogous to potential.

2. 2ND deriv $\equiv$ curvature. Expect to develop a more rigorous notion of curvature that looks like $\partial^2 g$, with 20 independent components.

3. Size of inertial region is $\sim 1/\sqrt{\partial^2 g}$.

   ("Riemann normal coordinates" explicitly build spacetime with this form.)

Curvature & curved manifolds

A manifold with curvature is one on which initially parallel trajectories do not remain parallel.

Example: Surface of a sphere.

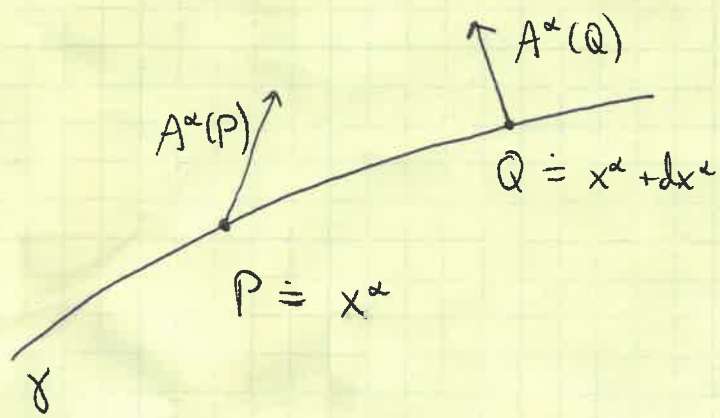
<u>NOT</u> an example: Surface of a cylinder.

We will think of vectors as residing in tangent space. Simplest to visualize as tangent plane to a point on a sphere.

Key complication of doing analysis in a curved manifold: Different tangent spaces for different points. Reflected in the fact that basis objects are functions, thus have nontrivial derivatives.

How do we take derivatives of a vector? Consider:

We have a curve γ in our manifold. γ has tangent vector $\vec{u}$. We want to differentiate a vector field $\vec{A}$ that is defined in the vicinity of γ:



$$\vec{u} = \frac{d\vec{x}}{d\lambda}$$

λ = affine parameter — specifies unit "length" along curve.

Our 1st guess: $\dfrac{\partial A^\alpha}{\partial x\beta} = \dfrac{A^\alpha(P) - A^\alpha(Q)}{dx^\alpha}$

Problem: P & Q don't have the same tangent space. This construction misses derivs of basis vectors. Can see that construction is not tensorial:

$$\partial_{\beta'} A^{\alpha'} \stackrel{?}{=} \frac{\partial x^{\alpha'}}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x\beta'} \partial_\beta A^\alpha$$

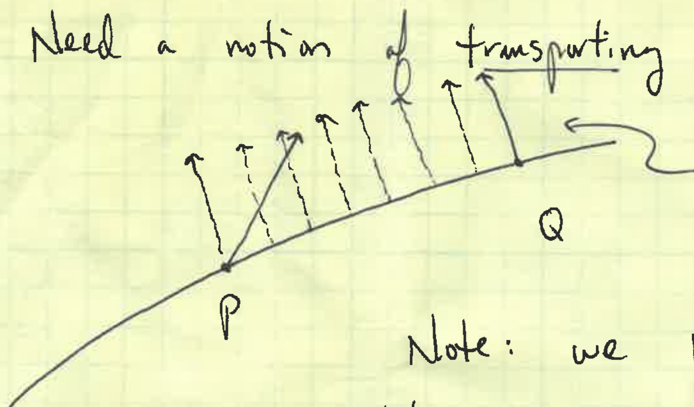
$$A^{\alpha'} = \frac{\partial x^{\alpha'}}{\partial x^\alpha} A^\alpha \quad , \quad \partial_{\beta'} = \frac{\partial x^\beta}{\partial x \beta'} \partial_\beta$$

$$\rightarrow \quad \partial_{\beta'} A^{\alpha'} = \frac{\partial x \beta}{\partial x \beta'} \partial_\beta \left(\frac{\partial x^{\alpha'}}{\partial x^\alpha} A^\alpha \right)$$

$$= \frac{\partial x \beta}{\partial x \beta'} \left(\frac{\partial x^{\alpha'}}{\partial x^\alpha} \partial_\beta A^\alpha + A^\alpha \frac{\partial^2 x^{\alpha'}}{\partial x \beta \, \partial x^\alpha} \right)$$

$\rightarrow$ Not simply linear in $\partial_\beta A^\alpha$.

Need a notion of transporting $A^\alpha(Q)$ to point P:



Parallel transport: Notion 1.

Note: we have implicitly used parallel transport all through physics! It's trivial in all previous applications since all points share tangent space in flat spacetime.

Need to specify a mechanism to transport the vector between these points. Let us assume an object exists which transports as follows:

$$A^{\alpha}_{PT}(P \to Q) = A^{\alpha}(P) - \Pi^{\alpha}_{\beta\mu}\, dx^{\beta} A^{\mu}$$

We make a derivative by comparing the vector at Q to the vector from P transported to Q:

$$D_{\beta} A^{\alpha} = \frac{A^{\alpha}(Q) - A^{\alpha}_{PT}(P \to Q)}{dx^{\beta}}$$

$$= \partial_{\beta} A^{\alpha} + \Pi^{\alpha}_{\beta\mu} A^{\mu}$$

"Connection"

If we demand that

$$\Pi^{\alpha'}_{\beta'\mu'} A^{\mu'} = \frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x^{\beta'}} \Pi^{\alpha}_{\beta\mu} A^{\mu}$$

$$- \frac{\partial^2 x^{\alpha'}}{\partial x^{\beta} \partial x^{\mu}} \frac{\partial x^{\beta}}{\partial x^{\beta'}} A^{\mu}$$

then $D_{\beta} A^{\alpha}$ will be tensorial: the "extra" term precisely cancels the term by which the partial derivative fails to be tensorial.

Further require $D_{\gamma}\, g_{\alpha\beta} = 0$ due to equivalence principle: True in LLF, true in all frames.

Last condition plus transformation property leads us to find

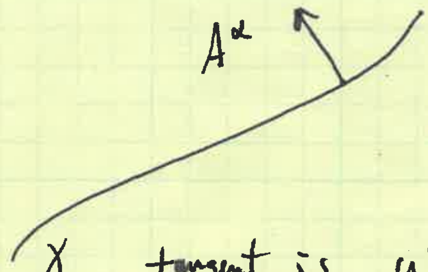
$$\Pi^{\alpha}{}_{\beta\mu} = \Gamma^{\alpha}{}_{\beta\mu}$$

$\rightarrow$ Connection is Christoffel

Therefore $\quad D_{\beta} = \nabla_{\beta}$

This derivative is the <u>covariant</u> derivative.

Why "parallel" transport? Imagine sliding along curve, using covariant derivative to define how to slide:



$$\frac{DA^\alpha}{d\lambda} \equiv u^\beta \nabla_\beta A^\alpha$$

"Parallel transport" is what we get when we slide along γ using the rule $DA^\alpha/d\lambda = 0$.

Let's examine what this means in the freely falling "local Lorentz frame":

$$u^\beta \nabla_\beta A^\alpha = u^\beta \left(\partial_\beta A^\alpha + \Gamma^\alpha{}_{\beta\mu} A^\mu \right)$$

But $\Gamma^\alpha{}_{\beta\mu} = 0$ in LLF, so the rule becomes

$$u^\beta \partial_\beta A^\alpha \equiv \frac{dA^\alpha}{d\lambda} = 0$$

→ We slide along γ keeping the components unchanged in the LLF.