Recap: Introduced notion of parallel transport, needed to compare vectors / tensors at different points in manifold.

Two approaches

1. Rigorous: Noting that naive partial derivative ~~introduces~~ contaminates result with terms that don't leave us with a tensorial result, we define a derivative operator with a "connection":

$$\nabla_\beta A^\alpha = \partial_\beta A^\alpha + \Pi^\alpha{}_{\beta\mu} A^\mu$$

Requirement that $\nabla_\beta g_{\mu\nu} = 0 \rightarrow \Pi^\alpha{}_{\beta\mu}$ is the Christoffel symbol defined earlier $\rightarrow$ This derivative is just the covariant derivative!

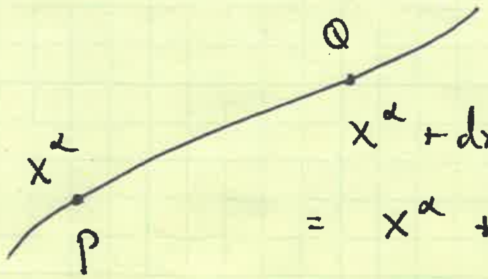
2. Physical: Go to LLF. Parallel transport reduces to just requiring components to slide along curve in manifold without change; find $\partial_\beta A^\alpha$ is good enough in LLF.

However: $\partial_\beta A^\alpha \equiv \nabla_\beta A^\alpha$ since partial & covariant are equivalent in LLF.

$\nabla_\beta A^\alpha$ is tensorial

What's tensorial in 1 frame is tensorial in all.

Another notion of transport:



$u^\alpha = dx^\alpha / d\lambda$, tangent on curve.

$$x^\alpha + dx^\alpha$$
$$= x^\alpha + u^\alpha d\lambda$$
$$\equiv (x')^\alpha$$

Now, regard shift from P to Q as a coordinate transformation:

$$A^\alpha_{LT}(P \to Q) = \text{vector } A^\alpha \text{ "transported" from P to Q}$$
$$= \frac{\partial (x')^\alpha}{\partial x^\beta} A^\beta$$
$$= \left(\delta^\alpha{}_\beta + (\partial_\beta u^\alpha)\, d\lambda\right) A^\beta(P)$$

$$\to A^\alpha_{LT}(P \to Q) = A^\alpha(P) + (\partial_\beta u^\alpha) A^\beta(P)\, d\lambda$$

The vector field at Q can be obtained by Taylor expansion:

$$A^\alpha(Q) = A^\alpha(x^\beta + dx^\beta)$$
$$= A^\alpha(x^\beta) + dx^\beta (\partial_\beta A^\alpha)\big|_P$$
$$= A^\alpha(P) + (u^\beta d\lambda)(\partial_\beta A^\alpha)$$

Define: $\mathcal{L}_{\vec{u}} A^\alpha \equiv \left(A^\alpha(Q) - A^\alpha_{LT}(Q)\right) / d\lambda$

$$= u^\beta \partial_\beta A^\alpha - A^\beta \partial_\beta u^\alpha$$

It's not hard to show that

$$\mathcal{L}_{\vec{u}} A^\alpha = u^\beta \nabla_\beta A^\alpha - A^\beta \nabla_\beta u^\alpha$$

The connection coefficients just cancel out.

This means that this object - "the Lie derivative of $\vec{A}$ along $\vec{u}$" - is itself tensorial.

Common notation: $\mathcal{L}_{\vec{u}} \vec{A} = [\vec{u}, \vec{A}]$

Repeat exercise for a scalar: $\mathcal{L}_{\vec{u}}\,\phi = u^\alpha \nabla_\alpha \phi$

1-form: Use $p_\alpha A^\alpha$ = scalar, take Lie of total, find

$$\mathcal{L}_{\vec{u}}\, p_\alpha = u^\beta \overset{\partial_\beta}{\nabla_\beta} p_\alpha + p_\beta \overset{\partial_\alpha}{\nabla_\alpha} u^\beta$$

Tensor: $$\mathcal{L}_{\vec{u}}\, T^\alpha{}_\beta = u^\mu \overset{\partial_\mu}{\nabla_\mu} T^\alpha{}_\beta - T^\mu{}_\beta \overset{\partial_\mu}{\nabla_\mu} u^\alpha + T^\alpha{}_\mu \overset{\partial_\beta}{\nabla_\beta} u^\mu$$

Utility: Consider a curve γ with tangent $\vec{u} = d\vec{x}/d\lambda$. A tensor is "Lie transported" along $\vec{u}$ if $\mathcal{L}_{\vec{u}}$ (tensor) $= 0$. (Used a lot in fluids: $\vec{u}$ defines <u>flow line</u>.)

If a tensor is Lie transported, then we can define the following coordinates: $x^0 = \lambda$ on the curve; x^1, x^2, x^3 = constant on the curve.

Then, $u^\alpha \equiv \dfrac{dx^\alpha}{d\lambda} \doteq \delta^\alpha{}_0 \rightarrow \partial_\mu u^\alpha = 0$!

Then, $\mathcal{L}_{\vec{u}}(\text{tensor}) = 0 \rightarrow \dfrac{\partial(\text{tensor})}{\partial x^0} = 0$

In these coordinates, the tensor is constant as we slide along the curve! Lie derivative gives us a covariant, frame independent way to describe a symmetry of a tensor field.

Example: suppose the tensor is the metric. Let us say there is some vector $\vec{\xi}$ such that the metric is Lie transported along $\vec{\xi}$:

$$\mathcal{L}_{\vec{\xi}}\, g_{\alpha\beta} = 0$$

1. There exists a coordinate x^0 such that $\frac{\partial g_{\alpha\beta}}{\partial x^0} = 0$:

   Metric is constant w.r.t. that coord. (Converse also holds: if the metric is constant wrt some coord, then a vector $\vec{\xi}$ exists such that the metric is Lie transported along $\vec{\xi}$.)

2. Expand the Lie derivative:

$$\mathcal{L}_{\vec{\xi}}\, g_{\alpha\beta} = 0$$

$$\rightarrow \quad \xi^\gamma \cancel{\nabla_\gamma g_{\alpha\beta}} + g_{\alpha\gamma} \nabla_\beta \xi^\gamma + g_{\beta\gamma} \nabla_\alpha \xi^\gamma = 0$$

$$\rightarrow \quad \boxed{\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0} \quad \text{"Killing's Eq"}$$

$\vec{\xi}$ is called a Killing vector - describes a symmetry of spacetime.

(Note: used fact that metric commutes with ∇_α)

Tensor densities: Quantities that transform almost, but not quite, like tensors. Off by a factor that looks like a determinant of a transformation matrix.

Most important tensor densities: Levi-Civita symbol & metric determinant.

Levi-Civita: $\tilde{\epsilon}_{\alpha\beta\gamma\delta} = +1$ for 0123 + even perms.

$= -1$ for odd perms

$= 0$ any index repeated.

Theorem: For any 4×4 matrix,

$$\tilde{\epsilon}_{\alpha\beta\gamma\delta} M^\alpha{}_\mu M^\beta{}_\nu M^\gamma{}_\rho M^\delta{}_\sigma = \tilde{\epsilon}_{\mu\nu\rho\sigma} |M|$$

Determinant of M.

Choose $M = \partial x^\mu / \partial x^{\mu'}$ - coord trans matrix:

$$\tilde{\epsilon}_{\alpha'\beta'\gamma'\delta'} = \left| \frac{\partial x^{\mu'}}{\partial x^\mu} \right| \tilde{\epsilon}_{\alpha\beta\gamma\delta} \frac{\partial x^\alpha}{\partial x^{\alpha'}} \frac{\partial x^\beta}{\partial x^{\beta'}} \frac{\partial x^\gamma}{\partial x^{\gamma'}} \frac{\partial x^\delta}{\partial x^{\delta'}}$$

<u>NOT</u> a tensor trans. law! Off by a factor of the Jacobian.

"Tensor density of weight 1."

Now look at metric:

$$g_{\alpha'\beta'} = \frac{\partial x^\alpha}{\partial x^{\alpha'}} \frac{\partial x^\beta}{\partial x^{\beta'}} g_{\alpha\beta}$$

$$\rightarrow \quad g' = \left| \frac{\partial x^{\alpha'}}{\partial x^\alpha} \right|^{-2} g$$

$\hookrightarrow = \det(g_{\alpha\beta})$.

Tensor density of weight -2.

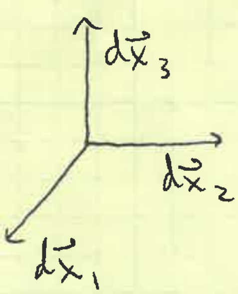
To convert a tensor density of weight w into a tensor, just multiply by $|g|^{w/2}$

$\hookrightarrow$ abs val!

Example: $\epsilon_{\alpha\beta\gamma\delta} \equiv \sqrt{|g|}\ \tilde{\epsilon}_{\alpha\beta\gamma\delta}$

$$\epsilon^{\alpha\beta\gamma\delta} = \frac{1}{\sqrt{|g|}} \tilde{\epsilon}^{\alpha\beta\gamma\delta}$$

This is an important example: use these tensors to form covariant volume operators:



$$dV^4 \equiv \sqrt{|g|}\ \tilde{\epsilon}_{\alpha\beta\gamma\delta}\, dx_0^\alpha\, dx_1^\beta\, dx_2^\gamma\, dx_3^\delta$$

$$= \sqrt{|g|}\ dx^0\, dx^1\, dx^2\, dx^3$$

if basis is orthogonal.

Intuition: In 3-D, spherical coords:

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

$$g_{\alpha\beta} \doteq \mathrm{diag}\,(1, r^2, r^2 \sin^2\theta)$$

$$\rightarrow \quad dV^3 = \sqrt{r^4 \sin^2\theta} \; dr \, d\theta \, d\phi$$

Determinant of metric <u>extremely</u> useful! Gives a shortcut for computing certain Christoffels:

$$\Gamma^\mu{}_{\mu\alpha} = g^{\mu\beta} \Gamma_{\beta\mu\alpha}$$

$$= \frac{1}{2} g^{\mu\beta} \left(\cancel{\partial_\mu g_{\alpha\beta}} + \partial_\alpha g_{\beta\mu} - \cancel{\partial_\beta g_{\mu\alpha}} \right)$$

(symm. of $g^{\mu\beta}$)

$$\rightarrow \quad \Gamma^\mu{}_{\mu\alpha} = \frac{1}{2} g^{\mu\beta} \partial_\alpha g_{\mu\beta}$$

To show: $$= \frac{1}{\sqrt{|g|}} \partial_\alpha \left(\sqrt{|g|} \right) \quad !$$

$$= \partial_\alpha \left(\ln \sqrt{|g|} \right)$$

Proof: Consider some matrix M. Consider the following variation:

$$\delta \ln(\det M) = \ln\left[\det(M + \delta M)\right] - \ln(\det M)$$

$$= \ln\left[\frac{\det(M + \delta M)}{\det M}\right]$$

$$= \ln\left[\det\left[1 + M^{-1} \cdot \delta M\right]\right]$$

Last line, used $(\det M)^{-1} = \det M^{-1}$ and

$$(\det M)(\det N) = \det(M \cdot N)$$

Useful identity: If ϵ is a "small" matrix, then

$$\det[1 + \epsilon] \simeq 1 + Tr(\epsilon) \quad \left(\text{where } Tr(\epsilon) = g^{\alpha\beta}\epsilon_{\alpha\beta} \equiv \epsilon^{\alpha}{}_{\alpha} \to \text{scalar.}\right)$$

So, $\delta \ln(\det M) = \ln\left[1 + Tr(M^{-1} \cdot \delta M)\right]$

$$\simeq Tr(M^{-1} \cdot \delta M)$$

Now $M \to g_{\alpha\beta}$: $\delta \ln|g| = Tr\left[g^{\mu\rho}\delta g_{\rho\gamma}\right]$

$$= g^{\mu\beta}\delta g_{\beta\mu}$$

$$\to \partial_\alpha \ln|g| = g^{\mu\beta}\partial_\alpha g_{\beta\mu}$$

$$\to \Gamma^{\mu}_{\mu\alpha} = \frac{1}{2} g^{\mu\beta}\partial_\alpha g_{\beta\mu}$$

$$= \partial_\alpha \ln|g|^{1/2}.$$

Utility: Divergences.

$$\nabla_\alpha A^\alpha = \partial_\alpha A^\alpha + \Gamma^\alpha{}_{\alpha\beta} A^\beta = \partial_\alpha A^\alpha + \Gamma^\beta{}_{\beta\alpha} A^\alpha$$

$$= \partial_\alpha A^\alpha + \frac{A^\alpha}{\sqrt{|g|}} \partial_\alpha \left(\sqrt{|g|}\right)$$

$$\longrightarrow \quad \nabla_\alpha A^\alpha = \frac{1}{\sqrt{|g|}} \partial_\alpha \left(\sqrt{|g|}\, A^\alpha\right)$$

Only involves partial derivatives!

Also gives a nice Gauss's Theorem:

$$\int_{V^4} \left(\nabla_\alpha A^\alpha\right) \sqrt{|g|}\, d^4x = \int \partial_\alpha \left(\sqrt{|g|}\, A^\alpha\right) d^4x$$

$$= \int_{\partial V^4} A^\alpha \sqrt{|g|}\, d\Sigma_\alpha .$$

Can we do this for tensors?

1. No: $\nabla_\alpha A^{\alpha\beta} = \partial_\alpha A^{\alpha\beta} + \Gamma^\alpha{}_{\alpha\gamma} A^{\gamma\beta} + \Gamma^\beta{}_{\alpha\gamma} A^{\alpha\gamma}$

   Nothing great unless $A^{\alpha\beta}$ is antisymmetric - then 2ND terms dies.

2. Not useful! $\nabla_\mu T^{\mu\gamma}$ is a 4-vector. Can't integrate this up to make a conservation law: comparing data at different tangent spaces.