How do we formulate the kinematics of bodies in curved spacetime?

Go to LLF, consider a "test mass": no charge, no spatial extent, no spin — just a pure point mass. Only "force" is gravity, which we no longer consider to be a force in the usual sense.

In this frame, trajectory is obvious: particle moves in a "straight line!" $x^\alpha = x_0^\alpha + u^\alpha \tau$

Cartesian

What does "straight" mean? Trajectory keeps going in the direction it started in: it parallel transports its tangent vector.

Consider trajectories parameterized by λ; $u^\alpha \equiv dx^\alpha / d\lambda$ is the tangent to the trajectory.

A curve which parallel transports its own tangent vector satisfies

$$u^\alpha \nabla_\alpha u^\beta = 0 \quad \text{or} \quad \nabla_{\vec{u}} \vec{u} = 0 \quad \text{or} \quad \frac{D\vec{u}}{d\lambda} = 0$$

Expand: $$u^\alpha \frac{\partial u^\beta}{\partial x^\alpha} + \Gamma^\beta{}_{\alpha\mu} u^\alpha u^\mu = 0$$

or
$$\boxed{\begin{aligned} &\frac{du^\beta}{d\lambda} + \Gamma^\beta{}_{\alpha\mu} u^\alpha u^\mu = 0 \\ &\frac{d^2x^\beta}{d\lambda^2} + \Gamma^\beta{}_{\alpha\mu} \frac{dx^\alpha}{d\lambda}\frac{dx^\mu}{d\lambda} = 0 \end{aligned}}$$

The "Geodesic Equation" ... curves known as geodesics.

Aside: more general form, move vector in a parallel fashion, but allow its normalization to change:

$$\frac{Du^\alpha}{d\lambda^*} = \kappa(\lambda^*)\, u^\alpha$$

Homework exercise: We can always reparameterize such that the RHS is zero:

Put $\quad v^\alpha = \dfrac{dx^\alpha}{d\lambda} \qquad v^\alpha \nabla_\alpha v^\beta = 0$

$\quad u^\alpha = \dfrac{dx^\alpha}{d\lambda^*} \qquad u^\alpha \nabla_\alpha u^\beta = \kappa(\lambda^*)\, u^\beta$

Then $$\frac{d\lambda}{d\lambda^*} = \exp\left[\int^\lambda \kappa(\lambda^*)\, d\lambda^*\right]$$

Parameterization with RHS = 0 is <u>affine</u> parameterization.

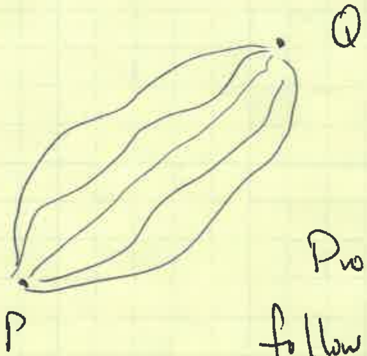
Intuition: Affine param. corresponds to uniformly spaced ticks on geodesic in the LLF. For timelike trajectories, very natural choice is proper time τ.

Reparameterization that leaves it affine:

$$\lambda' = a\lambda + b$$

a, b: constants. Shift origin, change tick spacing.

2ND path to geodesic eq: Based on rule "shortest path between 2 pts is a straight line"



Consider all curves which begin at P and end at Q.

Proper time elapsed by an observer who follows one of these paths is

$$\Delta\tau \equiv \int d\lambda \sqrt{-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}}$$

(Follows from $ds^2 = -d\tau^2$ in coords adapted to observer.)

Now, consider variations in path: put $f \equiv g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta$

Then $$\delta(\Delta\tau) = \int d\lambda \; \delta\left[(-f)^{1/2}\right]$$

$$= -\frac{1}{2}\int \left[-f\right]^{-1/2}\delta f \, d\lambda$$

Now, put $\lambda = \tau$: $\frac{dx^\alpha}{d\lambda} \rightarrow u^\alpha \rightarrow f = -1$.
(Timelike traj.)

$$\rightarrow \quad \delta(\Delta\tau) = -\frac{1}{2}\int \delta f \, d\tau$$

Let's now define

$$I \equiv \frac{1}{2}\int f \, d\tau = \frac{1}{2}\int \left(g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\right) d\tau$$

As an <u>action</u> for particle motion!

Now, vary the action: $x^\alpha \to x^\alpha + \delta x^\alpha$

$$g_{\alpha\beta} \to g_{\alpha\beta} + \delta x^\gamma \left(\partial_\gamma g_{\alpha\beta}\right)$$

with B.C. that $\delta x^\alpha = 0$ at endpoints.

$$\to \delta I = \frac{1}{2}\int d\tau \Big[\partial_\gamma g_{\alpha\beta} \frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\,\delta x^\gamma$$

$$+ g_{\alpha\beta}\frac{d(\delta x^\alpha)}{d\tau}\frac{dx^\beta}{d\tau} + g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{d(\delta x^\beta)}{d\tau}\Big]$$

Integration by parts on 2^{ND} term:

$$\frac{1}{2}\int d\tau \left[g_{\alpha\beta}\frac{d(\delta x^\alpha)}{d\tau}\frac{dx^\beta}{d\tau}\right] = -\frac{1}{2}\int d\tau \left[g_{\alpha\beta}\frac{d^2x^\beta}{d\tau^2} + \frac{dg_{\alpha\beta}}{d\tau}\frac{dx^\beta}{d\tau}\right]\delta x^\alpha$$

$$= -\frac{1}{2}\int d\tau \left[g_{\alpha\beta}\frac{d^2x^\beta}{d\tau^2} + \partial_\gamma g_{\alpha\beta}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau}\right]\delta x^\alpha$$

(Tossed a boundary term: $\delta x^\alpha = 0$ at boundaries.)

Likewise,

$$\frac{1}{2}\int d\tau \left[g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{d(\delta x^\beta)}{d\tau}\right] = -\frac{1}{2}\int d\tau \left[g_{\alpha\beta}\frac{d^2x^\alpha}{d\tau^2} + \partial_\gamma g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\gamma}{d\tau}\right]\delta x^\beta$$

Put all the pieces together:

$$\delta I = \frac{1}{2}\int d\tau \Big[\partial_\gamma g_{\alpha\beta}\, u^\alpha u^\beta \,\delta x^\gamma \ (1)$$

$$\overset{(2)}{-\,\partial_\gamma g_{\alpha\beta}\, u^\beta u^\gamma\, \delta x^\alpha} \ - \ \overset{(3)}{\partial_\gamma g_{\alpha\beta}\, u^\gamma u^\alpha\, \delta x^\beta}$$

$$\overset{(4)}{-\ g_{\alpha\beta}\frac{du^\beta}{d\tau}\,\delta x^\alpha} \ - \ g_{\alpha\beta}\frac{du^\alpha}{d\tau}\,\delta x^\beta \ (5) \Big]$$

Cycle dummy indices:

(1) Leave as is

(2) $\alpha \to \gamma$, $\beta \to \alpha$, $\gamma \to \beta$

(3) $\alpha \to \beta$, $\beta \to \gamma$, $\gamma \to \alpha$

(4) $\alpha \to \gamma$, $\beta \to \alpha$

(5) $\beta \to \gamma$

$$\to \delta I = -\int d\tau \Big[g_{\gamma\alpha}\frac{du^\alpha}{d\tau} + \underbrace{\frac{1}{2}\left(\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\gamma\alpha} - \partial_\gamma g_{\alpha\beta}\right)}_{\Gamma_{\gamma\alpha\beta}} u^\alpha u^\beta \Big]\delta x^\gamma$$

$$\to \quad \delta I = -\int d\tau \left[g_{\gamma\alpha}\frac{du^\alpha}{d\tau} + \Gamma_{\gamma\alpha\beta}\, u^\alpha u^\beta \right] \delta x^\gamma$$

Extremization: $\delta I = 0$ for <u>any</u> δx^γ

$\to$ Bracketed term $= 0$!

Hit with $g^{\mu\gamma}$:

$$\boxed{\frac{du^\mu}{d\tau} + \Gamma^\mu{}_{\alpha\beta} u^\alpha u^\beta = 0}$$

Same as before!

Recap: Geodesics generalize "straight" trajectories to curved spacetime.

Parallel transport: no force acts, so tangent moves parallel to itself.

Extremization: Geodesics generalize notion of "shortest distance between 2 pts."

Side note: Can rewrite in terms of momentum:

$$u^\alpha \nabla_\alpha u^\beta = 0 \quad \rightarrow \quad p^\alpha \nabla_\alpha p^\beta = 0$$

$$\rightarrow \quad m\frac{dp^\beta}{d\tau} + \Gamma^\beta{}_{\alpha\gamma}\, p^\alpha p^\gamma = 0$$

Useful rewriting: Define an affine parameter by

$$\Delta\lambda \equiv \Delta\tau / m$$

Then $\quad p^\alpha = dx^\alpha / d\lambda$

Momentum geodesic eq. reduces to "usual" one.

HOWEVER: Can take limit $m \rightarrow 0$, $\Delta\tau \rightarrow 0$, but $\Delta\tau / m \rightarrow$ constant. Gives us a geodesic equation that is suitable for null geodesics!

One further trick: Rewrite geodesic eq for downstairs momentum:

$$p^\alpha\left(\nabla_\alpha p_\beta\right) = 0$$

(ok: metric commutes with ∇_α)

$$\rightarrow \quad m\frac{dp_\beta}{d\tau} - \Gamma^\gamma_{\beta\alpha}\, p^\alpha p_\gamma = 0$$

$$\rightarrow \quad \frac{m\,dp_\gamma}{d\tau} = \Gamma_{\gamma\beta\alpha}\, p^\alpha p^\gamma = \frac{1}{2}\left(\partial_\beta g_{\alpha\gamma} + \partial_\alpha g_{\gamma\beta} - \partial_\gamma g_{\beta\alpha}\right) p^\alpha p^\gamma$$

antisym on $\alpha\gamma$ sym on $\alpha\gamma$

$$\rightarrow \quad \boxed{m\frac{dp_\beta}{d\tau} = \frac{1}{2}\,\partial_\beta\, g_{\alpha\gamma}\, p^\alpha p^\gamma}$$

Utility: if metric is independent of x^β, then p_β is a conserved constant on the geodesic!

Further, $\partial_\beta g_{\alpha\gamma} = 0 \rightarrow$ There exists a Killing vector ξ^β.

Examine how scalar $p^\beta \xi_\beta$ evolves on geodesic:

$$p^\alpha \nabla_\alpha\left(p^\beta \xi_\beta\right) = \xi_\beta \underbrace{\cancel{\left(p^\alpha \nabla_\alpha p^\beta\right)}}_{\text{geodesic eq}}{}^{\rightarrow 0} + p^\alpha p^\beta\, \nabla_\alpha \xi_\beta$$

$$= \frac{1}{2}\, p^\alpha p^\beta \left(\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha\right)$$

symmetry here lets us symmetrize here.

$$= 0 \quad \text{(Killing's Eq)}$$

$p^\beta \xi_\beta$ is constant on trajectory!

Examples in action:

$\partial_t g_{\alpha\beta} = 0 \rightarrow$ There exists a "timelike" Killing vector $\vec{\xi}^{T}$.

$\rightarrow \quad p_0 = -E$ Conserved energy!

$= -p^{\beta}\xi^{T}_{\beta}$ (coord. invariant expression)

$\partial_\phi g_{\alpha\beta} = 0 \rightarrow$ There exists an "axial" killing vector $\vec{\xi}^{\phi}$

$p_\phi \equiv L_z$

$= p^{\beta}\xi^{\phi}_{\beta}$.