Example geodesic: A spacetime we will soon derive,

$$ds^2 = -(1+2\phi)\,dt^2 + (1-2\phi)(dx^2+dy^2+dz^2)$$

$\phi \ll 1$ & $\phi = \phi(x,y,z) \rightarrow$ no time dependence.

Consider slow motion in the spacetime:

$$p^\alpha \doteq (E, \underline{p}) \qquad E \gg |\underline{p}| \; ; \; E \approx m$$

Geodesic equation: $$m\frac{dp^\beta}{d\tau} + \underbrace{\Gamma^\beta{}_{\mu\nu}\,p^\mu p^\nu}_{} = 0$$

$\hookrightarrow$ Dominated by $\mu=\nu=0$

$$\rightarrow \quad m\frac{dp^\beta}{d\tau} \simeq -\Gamma^\beta{}_{00}\,p^0 p^0 \simeq -m^2\,\Gamma^\beta{}_{00}$$

Focus on $\beta = i$: $$\Gamma^i{}_{00} = \frac{1}{2} g^{i\alpha}\,\Gamma_{\alpha 00}$$

$$= \frac{1}{2} g^{i\alpha}\left(\cancel{\partial_t g_{0\alpha}} + \cancel{\partial_t g_{\alpha 0}} - \partial_\alpha g_{00}\right)$$

$$g^{i\alpha} = (1-2\phi)^{-1}\,\delta^{i\alpha}$$

$$\rightarrow \Gamma^i{}_{00} = -\frac{1}{2}(1-2\phi)^{-1}\,\delta^{ij}\,\partial_j(-2\phi)$$

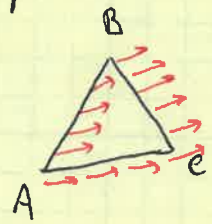
$$= \delta^{ij}\,\partial_j\phi + \mathcal{O}(\phi^2)$$

$$\rightarrow \boxed{\frac{dp^i}{d\tau} = -m\,\delta^{ij}\,\partial_j\phi}$$

ϕ is Newtonian gravitational potential: Motion in this spacetime is equivalent to motion under influence of Newtonian gravitational force!

Quantifying curvature: Curvature tells us about the breakdown of parallelism: Two initially parallel trajectories do not remain parallel. Will manifest as "geodesic deviation".

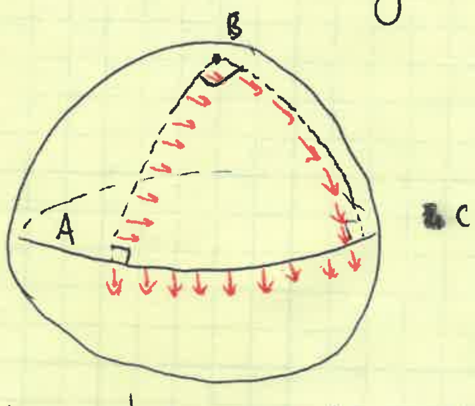
Begin with something simpler: Parallel transport of a vector around an equilateral triangle. 1st, triangle drawn on a flat surface:



$180^\circ \equiv$ sum of angles.

Nothing special: vector returns to start just like its initial configuration.

Now, "embed" triangle on the surface of a sphere:



$270^\circ \equiv$ sum of angles

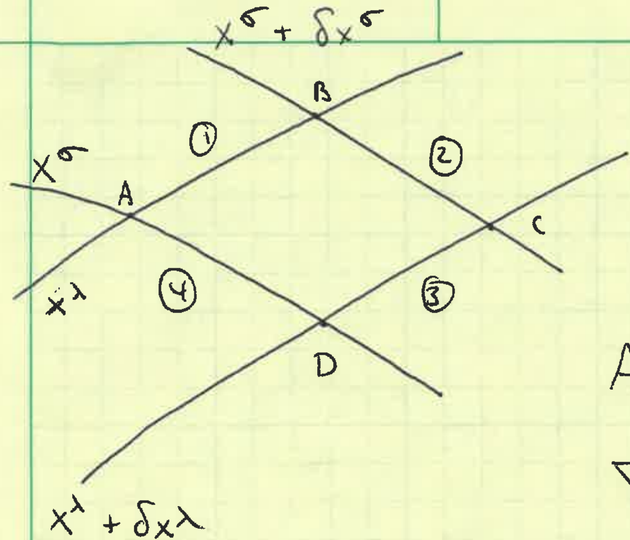
Vector returns rotated by

$90^\circ = (270 - 180)^\circ$!

Use this operation to quantify curvature: the shift of a vector after making a closed circuit.

This operation known as a holonomy:

http://mathworld.wolfram.com/HolonomyGroup.html



Consider parallel transport of a vector V^α around this loop:

$$A \to B \to C \to D \to A.$$

$A \to B$: Parallel transport tells us

$$\nabla_{\vec{e}_\sigma} \vec{V} = 0 \longrightarrow \partial_\sigma V^\alpha + \Gamma^\alpha_{\sigma\mu} V^\mu = 0$$

$$\rightarrow \frac{\partial V^\alpha}{\partial x^\sigma} = -\Gamma^\alpha_{\sigma\mu} V^\mu$$

Use this to integrate up to find the change as $A \to B$:

$$V^\alpha(B) = V^\alpha_{init} - \int_{①} \Gamma^\alpha_{\sigma\mu} V^\mu \, dx^\sigma$$

Likewise,

$$V^\alpha(C) = V^\alpha(B) - \int_{②} \Gamma^\alpha_{\lambda\mu} V^\mu \, dx^\lambda$$

$$\left.\begin{aligned} V^\alpha(D) &= V^\alpha(C) + \int_{③} \Gamma^\alpha_{\sigma\mu} V^\mu \, dx^\sigma \\ V^\alpha_{final} &= V^\alpha(D) + \int_{④} \Gamma^\alpha_{\lambda\mu} V^\mu \, dx^\lambda \end{aligned}\right\} \text{sign switch: coord decreases.}$$

Now, evaluate change:

$$\delta V^\alpha = V^\alpha_{final} - V^\alpha_{init}$$

$$= \int_{④} \Gamma^\alpha_{\lambda\mu} V^\mu dx^\lambda - \int_{②} \Gamma^\alpha_{\lambda\mu} V^\mu dx^\lambda \leftarrow$$

$$+ \int_{③} \Gamma^\alpha_{\sigma\mu} V^\mu dx^\sigma - \int_{①} \Gamma^\alpha_{\sigma\mu} V^\mu dx^\sigma \leftarrow$$

Each line represents "parallel" paths, slightly offset.

Integral ① evaluated along constant x^λ
③ " " " $x^\lambda + \delta x^\lambda$
② " " " $x^\sigma + \delta x^\sigma$
④ " " " x^σ

FIX

$$\rightarrow \int_{4} (\ ) \, dx^\lambda - \int_{2} (\ ) \, dx^\lambda \simeq - \delta x^\sigma \int_{2} \frac{\partial}{\partial x^\sigma} (\ ) \, dx^\lambda$$

$$\int_{3} (\ ) \, dx^\sigma - \int_{1} (\ ) \, dx^\sigma \simeq + \delta x^\lambda \int_{1} \frac{\partial}{\partial x^\lambda} (\ ) \, dx^\sigma$$

So,

$$\delta V^\alpha \simeq \int_{x^\sigma}^{x^\sigma + \delta x^\sigma} \delta x^\lambda \frac{\partial}{\partial x^\lambda} \left(\Gamma^\alpha_{\sigma\mu} V^\mu \right) dx^\sigma - \int_{x^\lambda}^{x^\lambda + \delta x^\lambda} \delta x^\sigma \frac{\partial}{\partial x^\sigma} \left(\Gamma^\alpha_{\lambda\mu} V^\mu \right) dx^\lambda$$

$$\simeq \delta x^\lambda \delta x^\sigma \left[\partial_\lambda \Gamma^\alpha_{\sigma\mu} V^\mu - \partial_\sigma \Gamma^\alpha_{\lambda\mu} V^\mu + \Gamma^\alpha_{\sigma\mu} \partial_\lambda V^\mu - \Gamma^\alpha_{\lambda\mu} \partial_\sigma V^\mu \right]$$

$$\partial_\lambda V^\mu = - \Gamma^\mu_{\lambda\nu} V^\nu$$

$$\partial_\sigma V^\mu = - \Gamma^\mu_{\sigma\nu} V^\nu$$

$$\rightarrow \delta V^\alpha = \delta x^\lambda \delta x^\sigma \Big[\left(\partial_\lambda \Gamma^\alpha_{\sigma\mu} - \partial_\sigma \Gamma^\alpha_{\lambda\mu} \right) V^\mu + \left(\Gamma^\alpha_{\lambda\mu} \Gamma^\mu_{\sigma\nu} - \Gamma^\alpha_{\sigma\mu} \Gamma^\mu_{\lambda\nu} \right) V^\nu \Big]$$

Relabel $\mu \leftrightarrow \nu$ on last two terms:

$$\delta V^\alpha \equiv \delta x^\lambda \delta x^\sigma V^\mu R^\alpha{}_{\mu\lambda\sigma}$$

where

$$\boxed{R^\alpha{}_{\mu\lambda\sigma} \equiv \partial_\lambda \Gamma^\alpha_{\sigma\mu} - \partial_\sigma \Gamma^\alpha_{\lambda\mu} + \Gamma^\alpha_{\lambda\nu} \Gamma^\nu_{\sigma\mu} - \Gamma^\alpha_{\sigma\nu} \Gamma^\nu_{\lambda\mu}}$$

"The Riemann curvature tensor"

Index ordering as in Carroll. Also agrees with MTW despite exchange on connection.

NOTE: T<u>ruly</u> tensorial, despite construction from Γs!

Equivalent definition: commutator of covariant derivatives

$$[\nabla_\lambda, \nabla_\sigma] V^\alpha = R^\alpha{}_{\mu\lambda\sigma} V^\mu$$

$$[\nabla_\lambda, \nabla_\sigma] p_\alpha = - R^\mu{}_{\alpha\lambda\sigma} p_\mu$$

Last one is <u>WRONG</u> in Schutz! (p171)

"Curvature coupling": When we examine processes that occur over an extended region, the curvature tensor plays an important role.

Discussion of geodesics strictly holds only for test bodies: zero size monopoles (that don't generate gravity).

More general bodies: motion involves coupling to Riemann. Essentially, the body has enough extent that different points in the body have different tangent spaces.

Example: Earth's extent couples to Riemann tensors of the sun + moon … changes the motion slightly, get precession of equinoxes.

Another example: Spin angular momentum constitutes "structure" beyond monopole, couples to curvature. Find equation of motion

$$u^\alpha \nabla_\alpha u^\beta \sim R^\beta{}_{\mu\nu\sigma} u^\mu S^{\nu\sigma}$$

$$S^{\nu\sigma} \sim \epsilon_{\alpha\beta}{}^{\nu\sigma} u^\alpha S^\beta$$

↳ spin vector.

Number of components: Naively n^4 - 256 in spacetime.

Symmetries: One we can see physically - transport in the opposite direction, switch sign of δV^α. Amounts to swapping indices λ and σ:

$$R^\alpha{}_{\mu\lambda\sigma} = - R^\alpha{}_{\mu\sigma\lambda}$$

Reduces number of components to $n^2 \cdot (n(n-1))/2 = 96$.

To get additional symmetries, lower the 1ST index,

$$R_{\alpha\mu\lambda\sigma} = g_{\alpha\nu} R^\nu{}_{\mu\lambda\sigma}$$

then, expand everything out by going into LLF - 1ST deriv of metric vanishes, so lose Christoffels; but, 2ND deriv does <u>not</u> vanish. NOTE: this is <u>only</u> for convenience - can do this exercise in full glory of Riemann, just a bit uglier.

LLF:
$$R_{\alpha\mu\lambda\sigma} = g_{\alpha\nu}\left[\partial_\lambda \Gamma^\nu{}_{\sigma\mu} - \partial_\sigma \Gamma^\nu{}_{\lambda\mu}\right]$$
$$= \partial_\lambda \Gamma_{\alpha\sigma\mu} - \partial_\sigma \Gamma_{\alpha\lambda\mu}$$

Used metric commutes with partial deriv in LLF. Note $R_{\alpha\mu\lambda\sigma} = - R_{\alpha\mu\sigma\lambda}$ is obvious here!

Now, insert definition of christoffels:

$$\left(R_{\alpha\mu\lambda\sigma}\right)^{LLF} = \frac{1}{2}\Big(\partial_\lambda \partial_\mu g_{\alpha\sigma} - \partial_\lambda \partial_\alpha g_{\sigma\mu}$$
$$- \partial_\sigma \partial_\mu g_{\alpha\lambda} + \partial_\sigma \partial_{\cancel{\sigma}\alpha} g_{\lambda\mu}\Big)$$

After staring for a while, "notice" the following symmetries:

1. $R_{\alpha\mu\lambda\sigma} = - R_{\alpha\mu\sigma\lambda}$ (already knew this one)

2. $R_{\alpha\mu\lambda\sigma} = - R_{\mu\alpha\lambda\sigma}$

3. $R_{\alpha\mu\lambda\sigma} = R_{\lambda\sigma\alpha\mu}$

4. $R_{\alpha\mu\lambda\sigma} + R_{\alpha\lambda\sigma\mu} + R_{\alpha\sigma\mu\lambda} = 0$

4.' $R_{\alpha[\mu\lambda\sigma]} = 0 \rightarrow$ totally equivalent. Expand out the full antisymmetrization:

$$R_{\alpha[\mu\lambda\sigma]} = \frac{1}{3!}\Big(R_{\alpha\mu\lambda\sigma} - R_{\alpha\lambda\mu\sigma} + R_{\alpha\lambda\sigma\mu}$$
$$- R_{\alpha\mu\sigma\lambda} + R_{\alpha\sigma\mu\lambda} - R_{\alpha\sigma\lambda\mu}\Big)$$

Use symmetry 1 a few times, recover other form of #4.

Careful counting: Syms 1-4 reduce the number of independent components:

$$n^4 \longrightarrow \frac{n^2(n^2-1)}{12}$$

Interesting cases:

$n=1 \rightarrow$ Riemann has NO independent components!
No holonomy in 1-D.

$n=2 \rightarrow \dfrac{n^2(n^2-1)}{12} = 1.$ Single number characterizes curvature $\rightarrow$ radius.

$n=4 \rightarrow \dfrac{n^2(n^2-1)}{12} = \dfrac{16 \cdot 15}{12} = 20$

$\rightarrow$ exactly the number of derivatives we could NOT cancel when we proved the existence of the LLF!

Fermi-Normal coordinates: Let γ be a geodesic curve. Let t be proper time on that geodesic, and let x^i be cartesian coordinates near it; $x^j = 0$ on the geodesic.

Then, in the LLF,

$$g_{tt} = -1 - R_{tjtk}\, x^j x^k + \mathcal{O}(x^3)$$

$$g_{tj} = -\frac{2}{3} R_{tijk}\, x^i x^k + \mathcal{O}(x^3)$$

$$g_{jk} = \delta_{jk} - \frac{1}{3} R_{jikl}\, x^i x^l + \mathcal{O}(x^3)$$

Poisson, "A relativist's toolkit", Sec 1.11.