Variants of the curvature tensor:

1. Take the trace on the 1ST + 3RD indices:

$$R^{\alpha}{}_{\mu\alpha\nu} \equiv g^{\alpha\beta} R_{\beta\mu\alpha\nu}$$
$$\equiv R_{\mu\nu}$$

The <u>Ricci</u> curvature tensor.

Easy to show this is symmetric on μ + ν:

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha{}_{\mu\nu} - \underbrace{\partial_\nu \Gamma^\alpha{}_{\alpha\mu}}_{\partial_\nu \partial_\mu (\ln|g|^{1/2})} + \Gamma^\alpha{}_{\alpha\beta}\Gamma^\beta{}_{\nu\mu} - \Gamma^\alpha{}_{\nu\beta}\Gamma^\beta{}_{\alpha\mu}$$

→ 10 components in 4D - sort of like "half" of the Riemann tensor.

Trace on any other pair of indices is either zero or related using symmetries.

1'. Trace of Ricci: $g^{\mu\nu} R_{\mu\nu} \equiv R = R^\mu{}_\mu$.

"Ricci scalar" or "curvature scalar".

Totally describes curvature in 2-D.

2. Ricci is "trace Riemann" ... "rest" of Riemann is the trace-free part: the Weyl curvature.

$$C_{\alpha\mu\lambda\sigma} \equiv R_{\alpha\mu\lambda\sigma} - \frac{2}{n-2}\left(g_{\alpha[\lambda}R_{\sigma]\mu} - g_{\mu[\lambda}R_{\sigma]\alpha}\right) + \frac{2}{(n-2)(n-1)}\, g_{\alpha[\lambda}\, g_{\sigma]\mu}\, R$$

- Only defined for $n \geq 3$

- Has the same symmetries as Riemann, but designed such that trace on any pair of indices is zero.

Number of independent components:

$$N_{\text{Weyl}} = \underbrace{\frac{n^2(n^2-1)}{12}}_{\text{Riemann}} - \underbrace{\frac{n(n+1)}{2}}_{\text{Ricci}}$$

$$= \frac{n(n^3 - 7n - 6)}{12}$$

$N_{\text{Weyl}}(3D) = 0$

$N_{\text{Weyl}}(4D) = 10.$

Trick for Weyl: Conformal transformation

Conformal transformations are a local change of scale:

$$ds \to \Omega\, ds \equiv d\tilde{s}$$

↳ Scale factor: function of the spacetime.

Shows up in metric via

$$\tilde{g}_{\mu\nu} = \Omega^2(\vec{x})\, g_{\mu\nu}$$

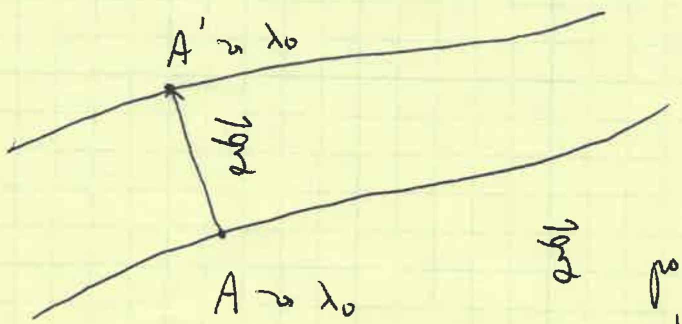
Often used because it leaves "light cones" invariant: if p^μ is null in the original metric, $g_{\mu\nu} p^\mu p^\nu = 0$, then $\tilde{g}_{\mu\nu} p^\mu p^\nu = \Omega^2 g_{\mu\nu} p^\mu p^\nu = 0$.

Curvature tensor (Riemann) gets horribly mangled by conformal transformation (see Carroll, Appendix G) ... but Weyl is invariant! Sometimes see Weyl called conformally invariant component of the curvature.

More physical (for us): Will soon see that Ricci is related to matter & energy content of spacetime; Weyl will then describe a fundamental "vacuum" curvature.

Breakdown of parallelism: Initially parallel trajectories (geodesics) become non-parallel. The geodesics <u>deviate</u> from their initial parallelism. (Yucky discussion: Carroll pp 145-146)

Consider two nearby geodesics, each parameterized by affine parameter λ:



$\gamma_{\vec{v}}$, tangent vector $\vec{v}$

$\gamma_{\vec{u}}$, tangent vector $\vec{u}$

$\vec{\xi}$ points from events at a particular λ on $\gamma_{\vec{u}}$ to events at that same λ on the other, $\gamma_{\vec{v}}$:

$$\vec{\xi} = \vec{x}(\gamma_{\vec{v}}, \lambda) - \vec{x}(\gamma_{\vec{u}}, \lambda)$$

Finally, assume the curves begin parallel:

$$\vec{u}(\lambda_0) = \vec{v}(\lambda_0)$$

(Curves must be close enough that they share initial tangent space to 1st order.)

This implies $\left(u^\alpha \nabla_\alpha \xi^\beta\right)\Big|_{\lambda_0} = 0$

$\rightarrow$ use as a boundary condition.

Goal now: develop an "acceleration equation" for ξ^α by comparing geodesics along $\gamma_{\vec{u}}$ to that along $\gamma_{\vec{v}}$.

Make analysis simple & bring in some intuition by starting in a LLF constructed centered on A. Not necessary! Just nice to clean things up.

Note: $g_{\mu\nu}|_A = \eta_{\mu\nu}$ $\quad \Gamma^\mu{}_{\alpha\beta}|_A = 0$

$g_{\mu\nu}|_{A'} = \eta_{\mu\nu}$ $\quad \Gamma^\mu{}_{\alpha\beta}|_{A'} = \partial_\gamma \Gamma^\mu{}_{\alpha\beta}|_A \, \xi^\gamma$

Geodesic equation for $\gamma_{\vec{u}}$ at A:

$$\left.\frac{d^2 x^\alpha}{d\lambda^2}\right|_A = 0 \qquad (\text{no } \Gamma\text{s in LLF})$$

Geodesic equation for $\gamma_{\vec{v}}$ at A':

$$\left.\frac{d^2 x^\alpha}{d\lambda^2}\right|_{A'} + \left[\Gamma^\alpha{}_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}\right]_{A'} = 0$$

$$\Gamma^\alpha{}_{\mu\nu}\big|_{A'} = \left(\partial_\beta \Gamma^\alpha{}_{\mu\nu}\right)_A \xi^\beta$$

$$\left.\frac{dx^\mu}{d\lambda}\right|_{A'} = V^\mu = u^\mu \rightarrow \text{initially parallel}$$

$$\rightarrow \quad \left.\frac{d^2 x^\alpha}{d\lambda^2}\right|_{A'} = -\partial_\beta \Gamma^\alpha{}_{\mu\nu} u^\mu u^\nu \xi^\beta$$

Difference between geodesic equations gives us an acceleration of the difference:

$$\left.\frac{d^2 x^\alpha}{d\lambda^2}\right|_{A'} - \left.\frac{d^2 x^\alpha}{d\lambda^2}\right|_A = \frac{d^2 \xi^\alpha}{d\lambda^2} = -\partial_\beta \Gamma^\alpha{}_{\mu\nu} u^\mu u^\nu \xi^\beta$$

Not bad! However, not tensorial. E.g., $d/d\lambda = u^\alpha \partial_\alpha$ — want to re-express as much as possible using covariant derivatives, try to get something that will be invariant in all reference frames.

Re-express using covariant deriv along trajectory:

$$\frac{D}{d\lambda} \equiv u^\alpha \nabla_\alpha$$

$$\frac{D\xi^\alpha}{d\lambda} = u^\beta \nabla_\beta \xi^\alpha = u^\beta \partial_\beta \xi^\alpha + u^\beta \Gamma^\alpha_{\beta\mu} \xi^\mu$$

$$= \frac{d\xi^\alpha}{d\lambda} + \Gamma^\alpha_{\beta\mu} u^\beta \xi^\mu$$

Don't zero yet (LLF) — wait til we've done another derivative.

$$\frac{D^2\xi^\alpha}{d\lambda^2} = u^\gamma \nabla_\gamma \left(\frac{d\xi^\alpha}{d\lambda} + \Gamma^\alpha_{\beta\mu} u^\beta \xi^\mu \right)$$

1st two terms: $u^\gamma \nabla_\gamma$ applied to 1st term in paren.

Last three terms, $u^\gamma \nabla_\gamma$ applied to 2ND term in paren.

$$= \frac{d^2\xi^\alpha}{d\lambda^2} + \cancelto{\text{LLF}}{u^\gamma \Gamma^\alpha_{\gamma\mu}} \frac{d\xi^\mu}{d\lambda} + \left(u^\gamma \nabla_\gamma \Gamma^\alpha_{\beta\mu}\right) u^\beta \xi^\mu$$

$$+ \Gamma^\alpha_{\beta\mu} \underbrace{\cancel{\left(u^\gamma \nabla_\gamma u^\beta\right)}}_{\text{geodesic}} \xi^\mu$$

$$+ \Gamma^\alpha_{\beta\mu} u^\beta \cancelto{0}{\left(u^\gamma \nabla_\gamma \xi^\mu\right)} \quad \text{initially parallel.}$$

$$\rightarrow \frac{D^2\xi^\alpha}{d\lambda^2} = \frac{d^2\xi^\alpha}{d\lambda^2} + \partial_\gamma \Gamma^\alpha_{\beta\mu} u^\beta u^\gamma \xi^\mu + \mathcal{O}(\Gamma^2)$$

toss in LLF.

Combine with result for $d^2\xi^\alpha/d\lambda^2$:

$$\frac{D^2\xi^\alpha}{d\lambda^2} = \partial_\gamma \Gamma^\alpha{}_{\beta\mu} u^\beta u^\gamma \xi^\mu - \partial_\beta \Gamma^\alpha{}_{\mu\nu} u^\mu u^\nu \xi^\beta$$

Relabel dummies: $\beta \to \mu$, $\mu \to \gamma$, $\nu \to \beta$ on 2ND term:

$$\rightarrow \quad \frac{D^2\xi^\alpha}{d\lambda^2} = \left(\partial_\gamma \Gamma^\alpha{}_{\beta\mu} - \partial_\mu \Gamma^\alpha{}_{\gamma\beta}\right) u^\beta u^\gamma \xi^\mu$$

$$\boxed{\frac{D^2\xi^\alpha}{d\lambda^2} = R^\alpha{}_{\beta\gamma\mu} u^\beta u^\gamma \xi^\mu}$$

Equation of geodesic deviation: gives us a covariant – tensorial! – notion of the action of tides.

Recall Riemann is the action of the commutator of two derivatives: $[\nabla_\lambda, \nabla_\sigma] V^\alpha = R^\alpha{}_{\mu\lambda\sigma} V^\mu$

→ Equivalent to our holonomic definition.

generalized: $[\nabla_\lambda, \nabla_\sigma] F^\alpha{}_\beta = R^\alpha{}_{\mu\lambda\sigma} F^\mu{}_\beta - R^\mu{}_{\beta\lambda\sigma} F^\alpha{}_\mu$

Consider the following two relations:

(A) $[\nabla_\alpha, \nabla_\beta] \nabla_\gamma p_\delta = -R^\mu{}_{\gamma\alpha\beta} \nabla_\mu p_\delta - R^\mu{}_{\delta\alpha\beta} \nabla_\gamma p_\mu$

(B) $\nabla_\alpha [\nabla_\beta, \nabla_\gamma] p_\delta = \nabla_\alpha \left(-R^\mu{}_{\delta\beta\gamma} p_\mu \right)$

$= -p_\mu \nabla_\alpha R^\mu{}_{\delta\beta\gamma} - R^\mu{}_{\delta\beta\gamma} \nabla_\alpha p_\mu$

metric commutes w/ ∇_α $\quad = -p^\mu \nabla_\alpha R_{\mu\delta\beta\gamma} - R^\mu{}_{\delta\beta\gamma} \nabla_\alpha p_\mu$

Riemann symmetry. $\quad = -p^\mu \nabla_\alpha R_{\beta\gamma\mu\delta} - R^\mu{}_{\delta\beta\gamma} \nabla_\alpha p_\mu$

Now, antisymmetrize on α, β, γ. Consider LHS of (A):

$$
\begin{aligned}
[\nabla_{[\alpha}, \nabla_\beta] \nabla_{\gamma]} p_\delta &= \frac{1}{3!} \Big([\nabla_\alpha, \nabla_\beta] \nabla_\gamma + [\nabla_\beta, \nabla_\gamma] \nabla_\alpha + [\nabla_\gamma, \nabla_\alpha] \nabla_\beta \\
&\qquad - [\nabla_\alpha, \nabla_\gamma] \nabla_\beta - [\nabla_\gamma, \nabla_\beta] \nabla_\alpha - [\nabla_\beta, \nabla_\alpha] \nabla_\gamma \Big) p_\delta \\
&= \frac{1}{3!} \Big(\nabla_\alpha [\nabla_\beta, \nabla_\gamma] + \nabla_\beta [\nabla_\gamma, \nabla_\alpha] + \nabla_\gamma [\nabla_\alpha, \nabla_\beta] \\
&\qquad - \nabla_\alpha [\nabla_\gamma, \nabla_\beta] - \nabla_\beta [\nabla_\alpha, \nabla_\gamma] - \nabla_\gamma [\nabla_\beta, \nabla_\alpha] \Big) p_\delta \\
&= \nabla_{[\alpha} [\nabla_\beta, \nabla_{\gamma]}] p_\delta \\
&= \text{LHS of (B), antisymmetrized}
\end{aligned}
$$

Antisymmetrization makes the equations equal! LHSs are clear, now examine RHSs:

$$R^\mu{}_{[\gamma\alpha\beta]} \nabla_\mu p_\delta + R^\mu{}_{\delta[\alpha\beta} \nabla_{\gamma]} p_\mu$$ the same.

$$= p^\mu \nabla_{[\alpha} R_{\beta\gamma]\mu\delta} + R^\mu{}_{\delta[\beta\gamma} \nabla_{\alpha]} p_\mu$$

Zero by Riemann symmetry.

$$\rightarrow \quad p^\mu \nabla_{[\alpha} R_{\beta\gamma]\mu\delta} = 0$$

Holds for any p^μ; so, we find the Bianchi identity:

$$\boxed{\begin{gathered} \nabla_{[\alpha} R_{\beta\gamma]\mu\delta} = 0 \\ \nabla_\alpha R_{\beta\gamma\mu\delta} + \nabla_\beta R_{\gamma\alpha\mu\delta} + \nabla_\gamma R_{\alpha\beta\mu\delta} = 0 \end{gathered}}$$

Rock on.