Recap: Derived Bianchi identity for Riemann tensor:

$$\nabla_{[\alpha} R_{\beta\gamma]\mu\nu} = 0$$

-or- $$\nabla_\alpha R_{\beta\gamma\mu\nu} + \nabla_\beta R_{\gamma\alpha\mu\nu} + \nabla_\gamma R_{\alpha\beta\mu\nu} = 0$$

Contract on $g^{\beta\mu}$:

$$\rightarrow \quad \nabla_\alpha R_{\gamma\nu} + \nabla^\mu R_{\gamma\alpha\mu\nu} - \nabla_\gamma R_{\alpha\nu} = 0$$

Contract on $g^{\gamma\nu}$:

$$\rightarrow \quad \nabla_\alpha R - \nabla^\mu R_{\alpha\mu} - \nabla^\nu R_{\alpha\nu} = 0$$

-or- $$\nabla^\mu \left(R_{\alpha\mu} - \frac{1}{2} g_{\alpha\mu} R \right) = 0$$

-or- $$\nabla^\mu G_{\alpha\mu} = 0$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$: the "Einstein tensor"

Note: $$G^\mu{}_\mu = g^{\mu\nu} G_{\mu\nu} \equiv G$$
$$= R - \frac{1}{2} g^\mu{}_\mu R$$
$$= R - 2R = -R$$

Einstein is <u>trace-reversed</u> Ricci.

Now ready to make a theory of gravity.

2 ingredients go into this:

1. Principle of equivalence → "minimal coupling principle"

   - Take a law of physics valid in inertial coordinates in flat spacetime (or in LLF)
   - Write in a coordinate invariant, tensorial form
   - Assert that the Resulting law holds in curved spacetime.

Example: force-free motion, $\frac{d^2 x^\mu}{d\tau^2} = 0 \longrightarrow u^\alpha \nabla_\alpha u^\beta = 0$.

Identical in LLF ... but one on the right is tensorial, and thus holds generally.

Another: local conservation of energy & momentum:

$$\partial_\mu T^{\mu\nu} = 0 \longrightarrow \nabla_\mu T^{\mu\nu} = 0$$

2. A field equation that connects spacetime to sources of matter & energy.

To nail down details, we will require our theory to recover the Newtonian limit:

$$\nabla^2 \Phi = \delta^{ij} \partial_i \partial_j \Phi = 4\pi G \rho$$

$$\underset{\sim}{\nabla} \Phi \Rightarrow \delta^{ij} \partial_j \Phi = - \frac{d^2 x^i}{dt^2}$$

First, recover Newtonian limit: Tool is the geodesic equation.

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$$

Slow motion limit: $\frac{dx^0}{d\tau} = \frac{dt}{d\tau} \gg \frac{dx^i}{d\tau}$ for all i.

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{00}\left(\frac{dt}{d\tau}\right)^2 = 0$$

$$\Gamma^\mu{}_{00} = \frac{1}{2} g^{\mu\nu}\left(\partial_0 g_{\nu 0} + \partial_0 g_{0\nu} - \partial_\nu g_{00}\right)$$

Drop time derivatives to recover Newtonian limit:

$$\rightarrow \Gamma^\mu{}_{00} = -\frac{1}{2} g^{\mu\nu}\partial_\nu g_{00}$$

Finally, consider weak deviations from flat spacetime:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$\rightarrow g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \quad (+\,\mathcal{O}(h^2))$$

Drop.

$$\rightarrow \Gamma^\mu{}_{00} = -\frac{1}{2}\eta^{\mu\nu}\partial_\nu h_{00}$$

Note: Raise & lower indices with $\eta_{\mu\nu}$. General rule in "linearized theory" - corrections to this are higher order in h.

Motion in this limit is given by

$$\frac{d^2 t}{d\tau^2} = 0 \qquad \text{via} \qquad \partial_0 h_{00} = 0$$

$$\frac{d^2 x^i}{d\tau^2} = \frac{1}{2} \eta^{ij} \partial_j h_{00} \left(\frac{dt}{d\tau} \right)^2$$

$$\rightarrow \quad \boxed{\frac{d^2 x^i}{dt^2} = \frac{1}{2} \delta^{ij} \partial_j h_{00}}$$

Correspondance with the Newtonian limit makes us choose

$$\boxed{\begin{aligned} h_{00} &= -2\Phi \qquad \text{or} \\ g_{00} &= -(1 + 2\Phi) \end{aligned}}$$

Field equation:

Newton says $\eta^{ij}\partial_i\partial_j \Phi = 4\pi G\rho$

Not tensorial! $\rho =$ mass density

$\rightarrow$ energy density

$\rightarrow$ T_{00}

$\rightarrow$ a tensor component, not a tensor.

Want something in which $T_{\mu\nu}$ acts as the source:

(Tensor related to spacetime geometry) $= T_{\mu\nu}$

Expect this tensor to be (heuristically) 2 derivatives of the spacetime metric — curvature.

Another principle: $\nabla_\mu T^{\mu\nu} = 0$ (or $\nabla^\mu F_{\mu\nu} = 0$)

$\rightarrow$ Source is divergence free, so spacetime tensor must be so as well.

That picks out the Einstein tensor as the one we want:

$$G_{\mu\nu} = \kappa T_{\mu\nu}$$

or

$$R_{\mu\nu} = \kappa\left(T_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} T\right)$$

Reminiscent of Maxwell "automagically" enforcing source conservation:

$$\nabla_\nu F^{\mu\nu} = 4\pi J^\mu$$

$$\nabla_\mu \nabla_\nu F^{\mu\nu} = 0 \qquad \text{(sym - antisym)}$$

$$\Rightarrow \nabla_\mu J^\mu = 0.$$

"Just" need to fix K. Go to static, weak field limit, try to recover Newtonian field equation; begin with a static perfect fluid source:

$$T_{\mu\nu} = (\rho + P) u_\mu u_\nu + P g_{\mu\nu}$$
$$= \rho\, u_\mu u_\nu \qquad (\rho \gg P \text{ in non-rel limit})$$

Static fluid implies $u^\mu \doteq (u^0, 0, 0, 0)$

$$g_{\mu\nu} u^\mu u^\nu = -1 \rightarrow u^0 = 1 + \tfrac{1}{2} h_{00}$$
$$u_0 = -u^0 \qquad (\text{raise \& lower with } \eta_{\mu\nu})$$

Now, we have enough pieces to generate a component of the proposed field equation:

$$R_{00} = K\left(T_{00} - \tfrac{1}{2} g_{00} T\right)$$

$$T_{00} = \rho\, u_0 u_0 = \rho(1 + h_{00})$$

$$T = g^{\mu\nu} T_{\mu\nu} = \rho\, u^\mu u_\mu^* = -\rho$$

$$\rightarrow T_{00} - \tfrac{1}{2} g_{00} T = \rho(1 + h_{00}) - \tfrac{1}{2}(-1 + h_{00})(-\rho)$$
$$= \tfrac{1}{2}\rho + \tfrac{3}{2} h_{00}\rho = \tfrac{1}{2}\rho(1 + 3h_{00})$$
$$\simeq \tfrac{1}{2}\rho$$

$$\rightarrow R_{00} \simeq \tfrac{1}{2} K \rho \quad \text{in this limit.}$$

Now, build R_{00}:

$$R_{00} = R^{\mu}{}_{0\mu0} = R^{i}{}_{0i0} \quad \text{since } R^{0}{}_{000} = 0$$

← indices!

$$\rightarrow R_{00} = \partial_i \Gamma^{i}{}_{00} - \cancel{\partial_0 \Gamma^{i}{}_{00}} + \mathcal{O}(\Gamma^2)$$

static

$\hookrightarrow \mathcal{O}(h^2)$

$$= \tfrac{1}{2}\partial_i \left[g^{i\mu}\left(\partial_0 g_{\mu 0} + \partial_0 g_{0\mu} - \partial_\mu g_{00}\right)\right]$$

$$= -\tfrac{1}{2}\partial_i \left[\eta^{i\mu}\partial_\mu h_{00}\right]$$

$$= -\tfrac{1}{2}\delta^{ij}\partial_i\partial_j h_{00} \equiv -\tfrac{1}{2}\nabla^2 h_{00}$$

So: $R_{00} = \kappa\left(T_{00} - \tfrac{1}{2} g_{00} T\right)$

$$\rightarrow \nabla^2 h_{00} = -\kappa\rho$$

Previous exercise: Newtonian limit gives us $h_{00} = -2\Phi$

$$\rightarrow \nabla^2 \Phi = \frac{\kappa\rho}{2} \longrightarrow \kappa = 8\pi G$$

$$\boxed{G_{\mu\nu} = 8\pi G\, T_{\mu\nu}}$$

The Einstein Field Equation

Rather ad hoc derivation! Essentially saying "$T_{\mu\nu}$ should be my source; $T_{\mu\nu}$ is divergence free; the field should therefore be a 2-index divergence free curvature tensor."

Can imagine other ways of building field equations that are also covariant → way to extend or modify G.R. Experiment ultimately serves as the final arbeiter.

Next topic: Deriving field equation from action. Action principle gives us a systematic way to motivate modifications to the standard theory.

Also, note this choice is not unique! We picked $G_{\mu\nu}$ as the LHS motivated by the requirement that things be divergence free. But <u>ANY</u> divergence free tensor could work... e.g., the metric itself:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G\, T_{\mu\nu}$$

"Cosmological constant"

Define : $T^{\Lambda}_{\mu\nu} = -\frac{\Lambda}{8\pi G} g_{\mu\nu}$

→ Can regard Λ as a "source" rather than as the "geometry"!

In particular, $T^{\Lambda}_{\mu\nu}$ is a perfect fluid with

$$\rho = \frac{\Lambda}{8\pi G}, \quad P = \frac{-\Lambda}{8\pi G}$$

Such stress energy tensors arise in quantum field theory - represents a form of vacuum energy (isotropic, invariant to Lorentz transformations in the LLF). Originally noted by Y. Zel'dovich.

Final point: often work in units in which $G = 1$. Very convenient both theoretically and experimentally, since G is very hard to measure, but GM is not. (Example: $GM_\odot$ known to about 9 digits)

Using $G = 1$ & $c = 1$, mass, time, and length are all in the same units:

$\frac{G}{c^2}$: converts SI mass to length.

$$\frac{GM_\odot}{c^2} = 1.47 \text{ km}$$

$\frac{GM}{c^3}$: Converts SI mass to time

$$\frac{GM_\odot}{c^3} = 4.92 \ \mu\text{sec}$$

$\frac{G}{c^4}$: converts energy to length.

$$T_{\mu\nu} = \text{energy / density} = \frac{\text{energy}}{\text{length}^3}$$

$$\frac{G}{c^4} T_{\mu\nu} = (\text{length})^{-2}$$

$$= \text{curvature.}$$

$$\longrightarrow \quad G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

really small! Takes a lot of energy to bend spacetime.