Remainder of the course: solving + applying the Einstein field equation, $G_{\alpha\beta} = 8\pi G T_{\alpha\beta}$ to build physically interesting spacetimes.

In general, the EFEs are horribly complicated if they are regarded as field equations for the spacetime metric. Consider a perfect fluid:

$$G_{\alpha\beta} = \mathcal{D}^2_{\alpha\beta}(g_{\mu\nu})$$

Complicated 2^{ND} order, nonlinear differential operator

$$T_{\alpha\beta} = (\rho + P) u_\alpha u_\beta + P g_{\alpha\beta}$$

Depends on fluid properties

and spacetime metric.

Note: fluid 4-vel also depends on metric through $u^\alpha u^\beta g_{\alpha\beta} = -1$.

Moral: general solution to EFE is extraordinarily difficult.

Road to success: Consider

1. "Weak" gravity: Spacetime not too different from $\eta_{\alpha\beta}$.
2. Symmetric solutions: With perturbations, quite powerful.
3. Numerical solution: Just code up that puppy and attack!

First look at "weak" gravity $\rightarrow$ "linearized" GR.

We consider spacetimes that are <u>nearly</u> flat. In this case, we can choose coordinates such that

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$$

where $\|h_{\alpha\beta}\| \equiv$ the magnitude of non-zero components of $h_{\alpha\beta}$

$$\ll 1.$$

Such a coordinate system is called "nearly Lorentz coordinates":

As close to true Lorentz / globally inertial coordinates as possible.

(Could use other coordinates ... but these are particularly convenient.)

Consider coordinate transformation of metric. General rule is

$$g_{\bar{\mu}\bar{\nu}} = L^{\alpha}{}_{\bar{\mu}} L^{\beta}{}_{\bar{\nu}} \, g_{\alpha\beta}$$

where $L^{\alpha}{}_{\bar{\mu}} = \partial x^{\alpha} / \partial x^{\bar{\mu}}$.

In flat spacetime, we picked out a special category of coordinate transformations:

$$\eta_{\bar{\mu}\bar{\nu}} = \Lambda^{\alpha}{}_{\bar{\mu}} \Lambda^{\beta}{}_{\bar{\nu}} \, \eta_{\alpha\beta}$$

where $\Lambda^{\alpha}{}_{\bar{\mu}}$ are Lorentz transformations and <u>BOTH</u> $\eta_{\alpha\beta}$ and $\eta_{\bar{\mu}\bar{\nu}}$ are $\mathrm{diag}(-1,1,1,1)$ - flat spacetime is invariant under Lorentz transformations.

In general, Lorentz only has physical meaning in S.R. - not applicable or useful in general, except with a LLF.

Just for fun, apply Lorentz transformation to nearly flat spacetime:

$$g_{\bar{\mu}\bar{\nu}} = \Lambda^{\alpha}{}_{\bar{\mu}} \Lambda^{\beta}{}_{\bar{\nu}} (\eta_{\alpha\beta} + h_{\alpha\beta})$$

$$= \eta_{\bar{\mu}\bar{\nu}} + \Lambda^{\alpha}{}_{\bar{\mu}} \Lambda^{\beta}{}_{\bar{\nu}} h_{\alpha\beta}$$

$$\equiv \eta_{\bar{\mu}\bar{\nu}} + h_{\bar{\mu}\bar{\nu}}$$

The "background" - flat spacetime - is unchanged by the Lorentz transformation, but the "perturbation" $h_{\alpha\beta}$ transforms like any tensor field in special relativity.

Name: "Background Lorentz transformation" since the flat background respects Lorentz invariance.

Relationship $h_{\bar{\mu}\bar{\nu}} = \Lambda^{\alpha}{}_{\bar{\mu}} \Lambda^{\beta}{}_{\bar{\nu}} h_{\alpha\beta}$ tells us that a useful fiction is to regard $h_{\alpha\beta}$ as a simple tensor field residing in flat spacetime. Fiction since spacetime is really curved ... but a useful fiction for calculational convenience.

Other operations in linearized theory:

$$h^{\alpha\beta} = g^{\alpha\mu} g^{\beta\nu} h_{\mu\nu}$$

$$= \eta^{\alpha\mu} \eta^{\beta\nu} h_{\mu\nu} + \mathcal{O}(h^2)$$

Always drop these terms in linearized theory.

→ Indices are raised and lowered using the background metric - consistent with treatment of $h_{\mu\nu}$ as a tensor field in a flat background.

Using this, we can invert the metric:

$$g^{\alpha\beta} g_{\beta\gamma} = \delta^\alpha{}_\gamma \quad \rightarrow \text{definition!}$$

$$(\eta^{\alpha\beta} + m^{\alpha\beta})(\eta_{\beta\gamma} + h_{\beta\gamma}) = \delta^\alpha{}_\gamma$$

of the same order as $h_{\alpha\beta}$, must be determined.

$$\delta^\alpha{}_\gamma + h^\alpha{}_\gamma + m^\alpha{}_\gamma + \mathcal{O}(h^2) = \delta^\alpha{}_\gamma$$

$$\rightarrow m^\alpha{}_\gamma = - h^\alpha{}_\gamma$$

$$\rightarrow \boxed{g^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta}}$$

To linear order.

An important detail: Consider a coordinate shift, defined by

$$x^{\alpha'} = x^\alpha + \xi^\alpha(x^\beta)$$

(Note: lousy notation, abusing indices. Live with it! Not intended as a coordinate invariant relation, but rather just a connection between quantities in 2 specific coordinate systems.)

Coordinate transformation matrix: $L^{\alpha'}{}_\beta = \delta^\alpha{}_\beta + \partial_\beta \xi^\alpha$

Take to be $\ll 1$

Inverse: $L^\alpha{}_{\beta'} = \delta^\alpha{}_\beta - \partial_\beta \xi^\alpha + \mathcal{O}((\partial\xi)^2)$ "Infinitesimal" transformation.

Examine how the metric changes under this coordinate transformation:

$$g_{\mu'\nu'} = L^\alpha{}_{\mu'} L^\beta{}_{\nu'} g_{\alpha\beta}$$

$$= (\delta^\alpha{}_\mu - \partial_\mu \xi^\alpha)(\delta^\beta{}_\nu - \partial_\nu \xi^\beta)(\eta_{\alpha\beta} + h_{\alpha\beta})$$

$$= \eta_{\mu\nu} + h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu + \mathcal{O}(h\partial\xi), \ldots$$

Effect of coordinate transformation is to change the perturbation:

$$h_{\mu\nu} \rightarrow h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$$

Transformation of this kind is called a GAUGE TRANSFORMATION.

Closely analogous to $A_\mu \rightarrow A_\mu + \partial_\mu \Lambda$ in E+M.

Now, build some physics! Run metric through machinery to build curvature tensors - discard nonlinear terms, raise & lower indices with $\eta_{\alpha\beta}$.

$$R_{\mu\alpha\nu\beta} = \frac{1}{2}\left(\partial_\alpha\partial_\nu h_{\mu\beta} + \partial_\mu\partial_\beta h_{\alpha\nu} - \partial_\alpha\partial_\beta h_{\mu\nu} - \partial_\mu\partial_\nu h_{\alpha\beta}\right)$$

[Useful aside - What happens to Riemann under gauge transformation?

Compare Riemann computed from $h_{\alpha\beta}$ to that with $h_{\alpha\beta} - \partial_\alpha\xi_\beta - \partial_\beta\xi_\alpha$:

$$\delta R_{\mu\alpha\nu\beta} = \frac{1}{2}\Big(-\partial_\alpha\partial_\nu\partial_\mu\xi_\beta - \partial_\alpha\partial_\nu\partial_\beta\xi_\mu - \partial_\mu\partial_\beta\partial_\alpha\xi_\nu$$
$$- \partial_\mu\partial_\beta\partial_\nu\xi_\alpha + \partial_\alpha\partial_\beta\partial_\mu\xi_\nu + \partial_\alpha\partial_\beta\partial_\nu\xi_\mu$$
$$+ \partial_\mu\partial_\nu\partial_\alpha\xi_\beta + \partial_\mu\partial_\nu\partial_\beta\xi_\alpha\Big) = 0 \;!$$

Gauge change changes metric, but leaves curvature unchanged!

Very closely analogous to how gauge change in E+M changes potentials but leaves fields unchanged:

metric $\leftrightarrow$ potential

curvature $\leftrightarrow$ field.]

$$R_{\alpha\beta} = \eta^{\mu\nu} R_{\mu\alpha\nu\beta}$$
$$= \frac{1}{2}\left(\partial_\alpha\partial^\mu h_{\mu\beta} + \partial_\beta\partial^\mu h_{\mu\alpha} - \partial_\alpha\partial_\beta h - \Box h_{\alpha\beta}\right)$$

where $h = h^\mu{}_\mu = \eta^{\mu\nu} h_{\mu\nu}$

$\Box = \partial^\mu\partial_\mu = \eta^{\mu\nu}\partial_\mu\partial_\nu$ = wave operator in flat spacetime.

$$R = \eta^{\alpha\beta} R_{\alpha\beta} = \frac{1}{2}\left(\partial^\alpha \partial^\mu h_{\mu\alpha} + \partial^\alpha \partial^\mu h_{\mu\alpha} - 2\Box h\right)$$

$$= \partial^\alpha \partial^\mu h_{\mu\alpha} - \Box h .$$

$$\rightarrow \quad G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = R_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta} R$$

$$= \frac{1}{2}\Big(\partial_\alpha \partial^\mu h_{\mu\beta} + \partial_\beta \partial^\mu h_{\mu\alpha} - \partial_\alpha \partial_\beta h - \Box h_{\alpha\beta} + \eta_{\alpha\beta}\Box h - \eta_{\alpha\beta}\partial^\mu \partial^\nu h_{\mu\nu}\Big)$$

Sleight of hand lets us clean this up: Define

$$\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta} h$$

Significance: $\bar{h} = \eta^{\alpha\beta}\bar{h}_{\alpha\beta} = h - \frac{1}{2} h \eta^{\alpha\beta}\eta_{\alpha\beta} = -h$

$\bar{h}_{\alpha\beta}$ is "trace-reversed" metric perturbation.

Insert $h_{\alpha\beta} = \bar{h}_{\alpha\beta} + \frac{1}{2}\eta_{\alpha\beta} h$ into Einstein tensor:

$$G_{\alpha\beta} = \frac{1}{2}\Big(\partial_\alpha \partial^\mu \bar{h}_{\mu\beta} + \cancel{\frac{1}{2}\eta_{\mu\beta}\partial_\alpha\partial^\mu h} + \partial_\beta\partial^\mu \bar{h}_{\mu\alpha}$$

$$+ \cancel{\frac{1}{2}\eta_{\mu\alpha}\partial_\beta\partial^\mu h} - \cancel{\partial_\alpha\partial_\beta h} - \Box\bar{h}_{\alpha\beta} - \cancel{\frac{1}{2}\eta_{\alpha\beta}\Box h}$$

$$+ \cancel{\eta_{\alpha\beta}\Box h} - \eta_{\alpha\beta}\partial^\mu\partial^\nu \bar{h}_{\mu\nu} - \cancel{\frac{1}{2}\eta_{\alpha\beta}\eta_{\mu\nu}\partial^\mu\partial^\nu h}\Big)$$

$$\longrightarrow \quad G_{\alpha\beta} = \frac{1}{2}\left(\partial_\alpha\partial^\mu \bar{h}_{\mu\beta} + \partial_\beta\partial^\mu\bar{h}_{\mu\alpha} - \eta_{\alpha\beta}\partial^\mu\partial^\nu\bar{h}_{\mu\nu} - \Box\bar{h}_{\alpha\beta}\right)$$

Taking advantage of the fact that $G_{\alpha\beta}$ is just trace reversed $R_{\alpha\beta}$.

If we could set $\partial^\mu \bar{h}_{\mu\nu} = 0$, we could cleanup even more.

$$\partial^\mu \bar{h}_{\mu\nu} = 0 \rightarrow 4 \text{ conditions.}$$

$$\text{Gauge generators: } \xi_\nu \rightarrow 4 \text{ free functions.}$$

Can we use gauge freedom to kill off those annoying terms?

$$h^{\text{new}}_{\mu\nu} = h^{\text{old}}_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$$

$$\bar{h}^{\text{new}}_{\mu\nu} = \bar{h}^{\text{old}}_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu + \eta_{\mu\nu} \partial^\alpha \xi_\alpha$$

$$\partial^\mu \bar{h}^{\text{new}}_{\mu\nu} = \partial^\mu \bar{h}^{\text{old}}_{\mu\nu} - \Box \xi_\nu - \cancel{\partial_\nu \partial^\mu \xi_\mu} + \cancel{\partial_\nu \partial^\alpha \xi_\alpha}$$

If we chose our gauge generators ξ_ν such that

$$\Box \xi_\nu = \partial^\mu \bar{h}^{\text{old}}_{\mu\nu}$$

Then we have $\partial^\mu \bar{h}^{\text{new}}_{\mu\nu} = 0$! Solutions of this guaranteed to exist as long as $\bar{h}^{\text{old}}_{\mu\nu}$ is well-behaved (fine for weak gravity).

"Lorentz gauge" (or Lorenz gauge) in analogy with similar gauge in E&M.

$$\text{LG:} \quad \boxed{G_{\alpha\beta} = -\frac{1}{2} \Box \bar{h}_{\alpha\beta}}$$

EFE: $$\boxed{\Box \bar{h}_{\alpha\beta} = -16\pi G T_{\alpha\beta}}$$

Take source to be a static perfect fluid (hydrostatic equilibrium). Take it to be non-relativistic: $\rho \gg P$ (ie, $c^2 \gg v_{sound}^2$). Then,

$$T_{\alpha\beta} \simeq \rho u_\alpha u_\beta$$

$$\rightarrow T_{00} \simeq \rho \quad \text{(all others negligible.)}$$

The only nontrivial field equation component is

$$\Box \bar{h}_{00} = -16\pi G \rho$$

Source is static $\rightarrow$ field is static:

$$\nabla^2 \bar{h}_{00} = -16\pi G \rho$$

$$\bar{h}_{00} = -4\Phi \quad \leftarrow \text{Newtonian potential.}$$

All other $\bar{h}_{\mu\nu} = 0$.

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$$

$$\rightarrow h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h} \quad \text{Trace reverse twice!}$$

$$\bar{h} = \eta^{\mu\nu}\bar{h}_{\mu\nu} = -\bar{h}_{00} = +4\Phi$$

So,
$$h_{00} = -4\Phi - \left(\frac{1}{2}\right)(\eta_{00})(4\Phi) = -2\Phi$$
$$h_{11} = h_{22} = h_{33} = 0 - \frac{1}{2}\cdot 4\Phi = -2\Phi$$

$$\rightarrow \boxed{\begin{aligned} ds^2 &= (\eta_{\mu\nu} + h_{\mu\nu})\, dx^\mu dx^\nu \\ &= -(1 + 2\Phi)\, dt^2 + (1 - 2\Phi)(dx^2 + dy^2 + dz^2) \end{aligned}}$$

Pset 7: Correction due to rotation of body.