Last time, showed that the EFE for linearized gravity can be put into the form

$$\Box \bar{h}_{\alpha\beta} = -16\pi T_{\alpha\beta}$$

If we put our coordinates into Lorentz gauge: $\partial^\mu \bar{h}_{\mu\nu} = 0$.
For a static source, $\Box \rightarrow \nabla^2 = \delta^{ij}\partial_i\partial_j$ - simplifies considerably, not too hard to solve.

What about a non-static source? Use the fact that lin. EFE is of a form commonly encountered in classical field theory:

$$\Box f(t, \underset{\sim}{x}) = s(t, \underset{\sim}{x})$$

Linearity means we can solve this using method of Green's functions [Arfken, 3RD ed, Sec 16.5-16.6]:

Replace $s(t, \underset{\sim}{x})$ with delta function; assert that the solution in this case is some function G:

$$\Box G(t, \underset{\sim}{x}; t', \underset{\sim}{x}') = \delta(t-t')\,\delta(\underset{\sim}{x} - \underset{\sim}{x}')$$

"field point" "source point"

$$s(t, \underset{\sim}{x}) = \int dt' \int d^3x' \, s(t', \underset{\sim}{x}')\,\delta(t-t')\,\delta(\underset{\sim}{x} - \underset{\sim}{x}')$$

$$\rightarrow \quad f(t, \underset{\sim}{x}) = \int dt' \int d^3x' \, s(t', \underset{\sim}{x}')\, G(t, \underset{\sim}{x}; t', \underset{\sim}{x}')$$

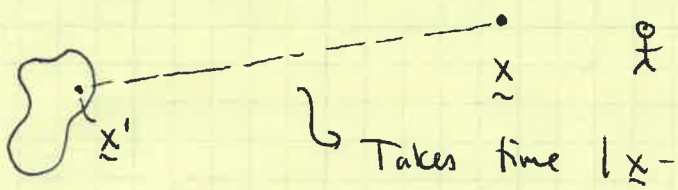
gives "field per unit source"

The Green's function for the wave equation is well-known (and easily computed via Fourier analysis):

$$G(t, \underset{\sim}{x}; t', \underset{\sim}{x}') = \frac{-\delta[t' - (t - |\underset{\sim}{x} - \underset{\sim}{x}'|)]}{4\pi |\underset{\sim}{x} - \underset{\sim}{x}'|}$$

(See, e.g., Jackson [2ND Ed], Sec 6.6)

Notice the form $t - |\underset{\sim}{x} - \underset{\sim}{x}'|$: This is "retarded time", the time at the field point minus the time it takes radiation to propagate from the source pt $\underset{\sim}{x}'$ to the field pt $\underset{\sim}{x}$:



Takes time $|\underset{\sim}{x} - \underset{\sim}{x}'|$ to propagate here.

Using this green's function, our Einstein solution is

$$\bar{h}_{\alpha\beta} = 4G \int d^3x' \, \frac{T_{\alpha\beta}(t - |\underset{\sim}{x} - \underset{\sim}{x}'|; \underset{\sim}{x}')}{|\underset{\sim}{x} - \underset{\sim}{x}'|}$$

Mild problem: <u>ALL</u> components of the metric look radiative!

This is because we choose a gauge which makes the field equation just a wave equation. Physical degrees of freedom could be masked by gauge effects!

Example: $A^0 = \frac{q}{r} - \frac{q\omega \sin(\underset{\sim}{k}\cdot\underset{\sim}{r} - \omega t)R}{r}$ $\quad r = \sqrt{x^2+y^2+z^2}$

$$A^i = -\frac{q k^i \sin(\underset{\sim}{k}\cdot\underset{\sim}{r} - \omega t)R}{r} - \frac{q x^i \cos(\underset{\sim}{k}\cdot\underset{\sim}{r} - \omega t)R}{r^3}$$

Looks radiative. Construct field tensor:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$\doteq \begin{bmatrix} 0 & -x & -y & -z \\ x & 0 & 0 & 0 \\ y & 0 & 0 & 0 \\ z & 0 & 0 & 0 \end{bmatrix} \times q/r^3$$

$\to E^i = q x^i / r^3, \quad B^i = 0$: Coulomb pt charge!

What happened? Picked a stupid gauge:

$$\Lambda = \frac{q\cos(\underset{\sim}{k}\cdot\underset{\sim}{r} - \omega t)R}{r}$$

$$A^\mu \to A^\mu + \partial^\mu \Lambda$$

Moral: Gauge can obscure physics if we're not careful.

Somewhat advanced topic: Recast metric & source in a form which allows us to categorize the radiative & non-radiative degrees of freedom of spacetime (in linearized theory, at least.)

End result: 4 physical degrees of freedom that are governed by ~~Laplace~~ Poisson-type equations

2 physical degrees of freedom governed by a wave equation: Polarizations of GWs.

Extension by Flanagan & Hughes of idea developed by Ed Bertschinger

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Consider $h_{\mu\nu}$ as a tensor field on a flat background. Pick inertial coordinates; choose timelike & spacelike directions. The 10 components of $h_{\mu\nu}$ then break into 3 subgroups when we consider how they behave under rotations:

$$h_{\mu\nu} \longrightarrow \quad h_{tt} \equiv -2\phi \quad \text{(scalar)}$$

$$h_{ti} \longrightarrow \text{vector}$$

$$h_{ij} \longrightarrow \text{tensor.}$$

Break down further. Consider h_{ti}: Any vector can be written as a divergence-free function plus the gradient of a scalar:

$$h_{ti} = \beta_i + \partial_i \gamma \quad , \quad \partial_i \beta_i = 0$$

Note: placement of spatial indices immaterial!

Next, consider h_{ij}. Extending our logic to a 2-index object, the most general form of this 3×3 symmetric tensor is

$$h_{ij} = h_{ij}^{TT} + \frac{1}{3} H \delta_{ij} + \partial_{(i} \varepsilon_{j)}$$
$$+ \left(\partial_i \partial_j - \frac{1}{3} \delta_{ij} \nabla^2\right) \lambda$$

(1) $H \equiv$ scalar. Notice: $\delta^{ij} h_{ij} = H$. This is the <u>trace</u> of h_{ij}.

(1) $\lambda \equiv$ scalar. This is the contribution to h_{ij} that is the double gradient of a scalar. Notice that the deriv. operator is traceless:

$$\delta^{ij}\left(\partial_i \partial_j - \frac{1}{3} \delta_{ij} \nabla^2\right) = \nabla^2 - \frac{1}{3} \cdot 3 \nabla^2 = 0.$$

Smart: H already has the trace.

(3-1) $\varepsilon_j \equiv$ vector. Gives us contribution that is the gradient of a vector. In order that it have no trace, we require $\partial_i \varepsilon_i = 0$.

(6-3-1) $h_{ij}^{TT} \equiv$ tensor. Gives us the remaining divergence free, trace-free degrees of freedom:

$$\delta^{ij} h_{ij}^{TT} = 0$$

$$\partial_i h_{ij}^{TT} = 0$$

$\rightarrow$ Total of 10 functions.

Goal: Develop EFE in terms of $(\phi, \beta_i, \gamma, H, \lambda, \varepsilon_i, h_{ij}^{TT})$

1st, examine gauge freedom. We can change the metric by $h_{\mu\nu} \to h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$ but leave all curvature unchanged.

Examine gauge generator: $\xi_\alpha \doteq (\xi_t, \xi_i)$

$$\doteq (A, B_i + \partial_i C) \quad w/ \ \partial_i B_i = 0$$

A: scalar

B_i: Divergence-free vector.

$\partial_i C$: Grad. of scalar

When we apply this to the metric, we find

$$\phi \to \phi + \partial_t A$$

$$\beta_i \to \beta_i - \partial_t B_i$$

$$\gamma \to \gamma - A - \partial_t C$$

$$H \to H - 2\nabla^2 C$$

$$\lambda \to \lambda - 2C$$

$$\varepsilon_i \to \varepsilon_i - 2\partial_t B_i$$

$$h_{ij}^{TT} \to h_{ij}^{TT} \leftarrow \text{gauge invariant D.O.F.!}$$

Stare at these; notice that the following combinations are gauge invariant:

$$\Phi = \phi + \partial_t \gamma - \frac{1}{2}\partial_t^2 \lambda$$

$$\Theta = \frac{1}{3}\left(H - \nabla^2 \lambda\right)$$

$$\Psi_i = \beta_i - \frac{1}{2}\epsilon_i \quad ; \quad \partial_i \Psi_i = 0$$

$$h_{ij}^{TT}$$

Only 6 functions! Of our original 10, only 6 are true physical degrees of freedom. Other 4 are pure gauge.

Consistent with 10 Einstein eqs plus 4 constraints $(\nabla^\alpha G_{\alpha\beta} = 0)$

Before apply this spacetime decomposition, need to decompose stress energy tensor:

$$T_{tt} \equiv \rho$$

$$T_{ti} \equiv S_i + \partial_i S$$

$$T_{ij} \equiv P\delta_{ij} + \sigma_{ij} + \partial_{(i}\sigma_{j)} + \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)\sigma$$

$\rho \equiv$ mass/energy density $\quad S_i \equiv$ momentum density / energy flux.

$P \equiv$ Pressure $\quad \sigma_{ij} \equiv$ anisotropic stress.

Constraints: $\partial_i S_i = 0 \qquad \partial_i \sigma_i = 0$

$\partial_i \sigma_{ij} = 0 \qquad \delta^{ij}\sigma_{ij} = 0$

No physics! Just rearranging terms.

Now, enforce $\partial^\alpha T_{\alpha\beta} = 0$; find

$$\nabla^2 S = \partial_t \rho$$
$$\nabla^2 \sigma = -\tfrac{3}{2} P + \tfrac{3}{2} \partial_t S$$
$$\nabla^2 \sigma_i = 2 \partial_t S_i$$

→ Only ρ, P, S_i, and σ_{ij} are freely specifiable: these 6 functions determine the remaining 4 fields.

Now, grind:

$$G_{tt} = -\nabla^2 \Theta$$

$$G_{ti} = -\tfrac{1}{2}\nabla^2 \Psi_i - \partial_i \partial_t \Theta$$

$$G_{ij} = -\tfrac{1}{2} \Box h^{TT}_{ij} - \partial_{(i} \Psi_{j)}$$
$$- \tfrac{1}{2} \partial_i \partial_j (2\Phi + \Theta)$$
$$+ \delta_{ij}\left(\tfrac{1}{2}\nabla^2(2\Phi + \Theta) - \partial_t^2 \Theta\right)$$

Equate to source:

$$\boxed{\begin{aligned} \nabla^2 \Phi &= 4\pi G(\rho + 3P - 3\partial_t S) \\ \nabla^2 \Theta &= -8\pi G \rho \\ \nabla^2 \Psi_i &= -16\pi G S_i \\ \Box h^{TT}_{ij} &= -16\pi G \sigma_{ij} \end{aligned}}$$

<u>REMARKABLE</u> cleanup! 10 EFEs reduce to 6 gauge invariant field equations. 4 are Poisson-type - yield solutions like Coulomb pt charge. Often called "longitudinal degrees of freedom"

2 are wave equations: "Radiative degrees of freedom"