Last lecture, we showed that - at leading order - the Einstein equations reduce to a single wave equation plus a few Poisson equations:

$$\nabla^2 \Phi = 4\pi G (\rho + 3P - \partial_t S)$$
$$\nabla^2 \Theta = -8\pi G \rho$$
$$\nabla^2 \Phi_i = -16\pi G S_i$$
$$\Box h_{ij}^{TT} = -16\pi G \sigma_{ij}$$

These 6 functions (recall constraints!) obey these equations in <u>all</u> gauges.

Focus today: The wave equation. Describes a propagating, radiative spacetime contributor - gravitational waves. Goals:

1. Understand nature of h_{ij}^{TT} in terms of observables.
2. See how to compute h_{ij}^{TT} given a source.

Begin: Just choose a metric perturbation with $\Phi = \Theta = \Phi_i = 0$.

Pick $h_{ij}^{TT} = h_{ij}^{TT}(t-z)$ - radiation propagating in $+z$ direction.

$$h_{\alpha\beta} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_{xx}^{TT} & h_{xy}^{TT} & 0 \\ 0 & h_{yx}^{TT} & h_{yy}^{TT} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\delta^{ij} h_{ij}^{TT} = 0 \rightarrow$

$h_{xx}^{TT} = -h_{yy}^{TT}$.

Symmetry: $h_{xy}^{TT} = h_{yx}^{TT}$.

Will justify this form later.

Consider the action of this metric on free particles:

Such particles move on geodesics,

$$\frac{du^\alpha}{d\tau} + \Gamma^\alpha{}_{\mu\nu} u^\mu u^\nu = 0$$

Particles are initially at rest in the coordinates, so

$$u^\alpha \big|_{\tau=0} \doteq (1, 0, 0, 0)$$

Only connection coefficient we care about is

$$\begin{aligned} \Gamma^\alpha{}_{00} &= \eta^{\alpha\beta}\,\Gamma_{\beta 00} \\ &= -\tfrac{1}{2}\,\eta^{\alpha\beta}\left(\partial_0 h^{TT}_{\beta 0} + \partial_0 h^{TT}{}_{0\beta} - \partial_\beta h^{TT}{}_{00}\right) \\ &= 0 \end{aligned}$$

Hence $\frac{du^\alpha}{d\tau} = 0$ at $\tau = 0$ ... integrate up, find particle remains FOREVER at rest!

So GWs have no effect? No, we've just asked a stupid question! Geodesic equation is a statement about motion with respect to coordinates. This result means that free particles are at rest IN THESE COORDINATES. The coordinates could themselves be moving!

Moral: Focus on "proper" quantities.

Consider 2 nearby particles that each follow geodesics:

origin   displaced from origin by $\varepsilon \vec{e}_x$

A   B

Consider time for a light pulse to travel from A to B:

$$0 = -\left(\frac{dt}{d\lambda}\right)^2 + \left(1 + h_{xx}^{TT}\right)\left(\frac{dx}{d\lambda}\right)^2$$

$$dt = \sqrt{1 + h_{xx}^{TT}}\, dx$$

$$\simeq \left(1 + \frac{h_{xx}^{TT}}{2}\right) dx$$

$$T_{A \to B} = \int_0^{\varepsilon} \left(1 + \frac{h_{xx}}{2}\right) dx$$

$$= \varepsilon + \frac{1}{2}\int_0^{\varepsilon} h_{xx}\, dx$$

Time of arrival of light pulses varies with the amplitude of this component of the GW.

More sophisticated treatment: Examine the geodesic separation of these two particles.

Take them to be (initially) close enough that they have the same 4-velocity:

$$\vec{u} = (1, \underset{\sim}{0}) + \mathcal{O}(h)$$

↳ Couples to produce $\mathcal{O}(h^2)$ corrections. Can ignore.

Equation of geodesic deviation:

$$\frac{D^2 \xi^\alpha}{d\tau^2} = R^\alpha{}_{\mu\nu\beta} u^\mu u^\nu \xi^\beta$$

becomes

$$\frac{\partial^2 \xi^i}{\partial t^2} = R^i{}_{00j}\, \xi^j + \mathcal{O}(h^2)$$

Now compute non-zero Riemann components for our perturbation:

$$R_{x0x0} = R^x{}_{0x0} = -\frac{1}{2} \partial_t^2 h_{xx}^{TT}$$

$$R_{y0y0} = R^y{}_{0y0} = -\frac{1}{2} \partial_t^2 h_{yy}^{TT} = +\frac{1}{2} \partial_t^2 h_{xx}^{TT}$$

$$R_{y0x0} = R^y{}_{0x0} = -\frac{1}{2} \partial_t^2 h_{xy}^{TT}$$

All others: zero, or related by symmetry.

Equation of geodesic deviation gives

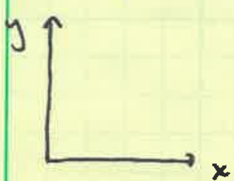
$$\partial_t^2 \xi^x = \frac{1}{2}\partial_t^2 h_{xx}^{TT}\xi^x + \frac{1}{2}\partial_t^2 h_{xy}^{TT}\xi^y$$

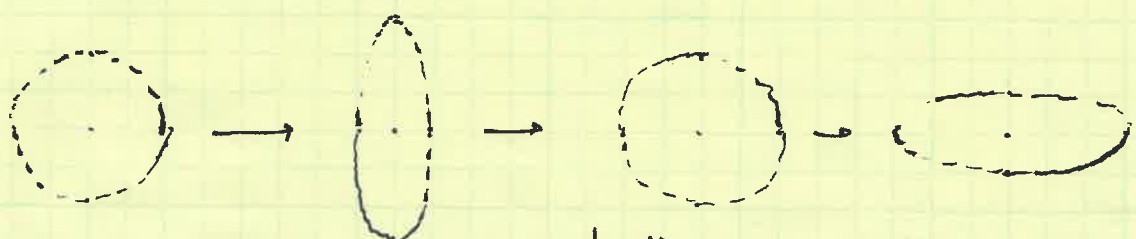
$$\partial_t^2 \xi^y = \frac{1}{2}\partial_t^2 h_{xy}^{TT}\xi^x - \frac{1}{2}\partial_t^2 h_{xx}^{TT}\xi^y$$

$$\partial_t^2 \xi^z = 0$$

Assume $h_{ij}^{TT} \ll 1$, and that $\xi^i = \xi^i_0 + \delta\xi^i$, with ξ^i_0 = constant and $\delta\xi^i \sim h$. Consider 2 limits:

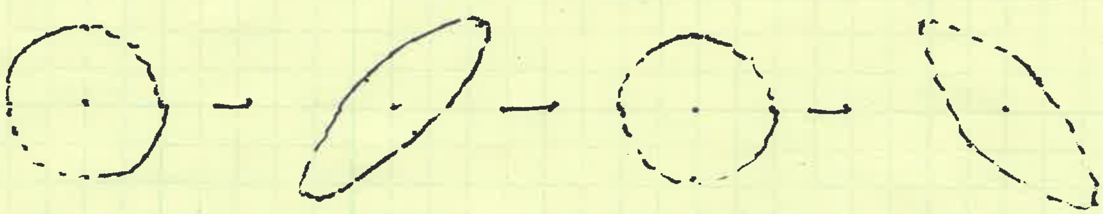
1ST, h_{xx}^{TT} = sinusoid, $h^{TT}_{xy} = 0$





"plus" polarization, h_+

2ND, h_{xy}^{TT} = sinusoid, $h_{xx}^{TT} = 0$



"Cross" polarization, $h_\times$

Action of wave is a tidal stretch & squeeze

Next: How to calculate radiation given a source.

Lesson of the gauge invariant formalism:

1. In some gauges (e.g., Lorentz) components of the metric appear radiative in that they satisfy a wave equation. Illusion of coordinate choice.

2. Only h^{TT}_{ij} satisfies wave equation in <u>ALL</u> gauges.

Useful Method: (a) Compute $h_{\alpha\beta}$ by <u>any</u> convenient method, in <u>any</u> convenient gauge.

(b) Project out the h^{TT}_{ij} subpiece of this.

Result is <u>guaranteed</u> to be the gauge invariant, physical radiation content of the spacetime!

Particularly convenient starting point: Linearized Einstein in Lorentz gauge:

$$\Box \bar{h}_{\alpha\beta} = -16\pi G T_{\alpha\beta}$$

$$\longrightarrow \quad \bar{h}_{\alpha\beta} = 4G \int d^3x' \, \frac{T_{\alpha\beta}(t - |\underset{\sim}{x} - \underset{\sim}{x}'| ; \underset{\sim}{x}')}{|\underset{\sim}{x} - \underset{\sim}{x}'|}$$

$\underset{\sim}{x} \equiv$ field point

$\underset{\sim}{x}' \equiv$ source point.

Typically interested in regime where $|\underset{\sim}{x} - \underset{\sim}{x}'| \gg$ size of source:

$\dot{\underset{\sim}{x}}$

Can then approximate:

$$\bar{h}_{\alpha\beta} \simeq \frac{4G}{r} \int d^3x' \, T_{\alpha\beta}(t-r; \underset{\sim}{x}')$$

More careful expansion uses the fact that $|\underset{\sim}{x} - \underset{\sim}{x}'|^{-1}$ can be written as a sum over Legendre polynomials - defines a multipole expansion.

Next, we only care about the spatial components:

$$\bar{h}_{ij} = \frac{4G}{r} \int d^3x' \, T_{ij}(t-r, \underset{\sim}{x}')$$

Invoke tensor virial theorem from pset 2:

$$\bar{h}_{ij} = \frac{2G}{r} \frac{\partial^2}{\partial t^2} \int d^3x' (T_{00}) \, x_i' x_j' \, .$$

$$\equiv \frac{2G}{r} \frac{\partial^2 I_{ij}}{\partial t^2}$$

Now, we just need to pull out the transverse + traceless piece of this.

Consider transverse condition first. Far from the source, we have $\Box \bar{h}_{ij} = 0$, which suggests we expand our solution in plane waves:

$$\bar{h}_{je} = A_{je}\, e^{i(\omega t - \underset{\sim}{k}\cdot\underset{\sim}{x})}$$

Amplitude A_{je} has no dependence on t or $\underset{\sim}{x}$; may depend on ω and $\underset{\sim}{k}$. (Should write this as an integral over ω & $\underset{\sim}{k}$ - just considering things mode-by-mode, which is good enough.)

Transversality: $$\partial^j \bar{h}_{je} = -i k^j A_{je}\, e^{i(\omega t - \underset{\sim}{k}\cdot\underset{\sim}{x})}$$

$\rightarrow$ $k^j \bar{h}_{je} = 0$ is the condition we want to enforce.

To guarantee that our solution satisfy this condition, define the projection tensor:

$$P_{ij} = \delta_{ij} - n_i n_j$$

Metric for the subspace orthogonal to the direction of propagation:

$n_i = k_i / \sqrt{\underset{\sim}{k}\cdot\underset{\sim}{k}}$ is a unit vector (or unit 1-form) along the direction of propagation.

Pset 1, Problem 3.

Now make the transverse perturbation:

$$\bar{h}^{T}_{ij} = \bar{h}_{ek} P_{ei} P_{kj}$$

Finally, remove trace. General formula for a tracefree 2-index tensor is

$$a_{ij}^{TF} = a_{ij} - \frac{1}{N} a_{kl} g^{kl} g_{ij}$$

where g_{ij} is the metric describing the manifold in which a_{ij} lives, and N is its dimension. ($g^{ij} g_{ij} = N$.)

In this case, $g_{ij} = P_{ij}$ and $N = 2$. So,

$$\bar{h}^{TT}_{ij} = h^{TT}_{ij} \leftarrow \text{no trace anymore}$$

$$= \bar{h}_{lk} P_{li} P_{kj} - \frac{1}{2} \bar{h}_{lk} P_{lk} P_{ij}$$

or

$$\boxed{h^{TT}_{ij} = \frac{2G}{r} \frac{d^2 I_{lk}}{dt^2} \left(P_{li} P_{kj} - \frac{1}{2} P_{lk} P_{ij} \right)}$$

The Quadrupole formula for GWs.