GWs to date only defined in context of linearized theory on a flat background. Useful, but restrictive!

general case: waves propagating on a reasonably well-characterized but non-flat background:

$$g_{\alpha\beta} \equiv \hat{g}_{\alpha\beta} + h_{\alpha\beta} \rightarrow \text{generally, but not necessarily small.}$$

spatially + temporally varying

How we define "wave" in this case? Local measurements can only determine $g_{\alpha\beta}$ (or some surrogate) - no clear way to distinguish in general background from radiation.

Trick is to use separation of lengthscales + timescales: GW is oscillatory; time/lengthscale over which wave varies is much shorter than those on which background varies.

Analogy: Water wave on ocean. What is wave, what is natural "curvature" due to bending of earth, local structures? Obvious when you see it, thanks to lengthscale separation.

We now assume at least two sets of natural length scales can be used in our analysis:

$(\mathcal{L}, \mathcal{T})$: "long" length + timescales on which background varies.

(λ, τ) : "short" scales, wavelength + period of GW.

We can then always remove the oscillation by averaging over $L \sim$ (several) λ, $T \sim$ (several) τ:

$$\hat{g}_{\alpha\beta} \equiv \langle g_{\alpha\beta} \rangle$$

The wave is found by subtracting: $h_{\alpha\beta} = g_{\alpha\beta} - \langle g_{\alpha\beta} \rangle$

Averaging procedure first made rigorous by Brill + Hartle: Phys Rev 135, 271 (1964).

$$\langle g_{\alpha\beta} \rangle = \int dV^4 \; g_{\alpha\beta} \, f(x^\mu)$$

Weighting, defined such that $\int dV^4 f(x^\mu) = 1$.

Peaked, but with width $\sim L$.

gives rigorously tensorial "output" up to terms $\mathcal{O}(\lambda/\mathcal{L})$.

Now that wave and background are defined, we can discuss linearized about general background:

$$g_{\alpha\beta} = \cancel{g^{B}_{\alpha\beta} + h_{\alpha\beta}} \equiv \hat{g}_{\alpha\beta} + h_{\alpha\beta}.$$

Develop all the geometric stuff, discarding $\mathcal{O}(h^2)$, being very careful about background.

Connection:

$$\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}\left(\partial_\mu g_{\nu\beta} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}\right)$$

$$= \frac{1}{2}\left(\hat{g}^{\alpha\beta} - h^{\alpha\beta}\right)\left(\partial_\mu \hat{g}_{\nu\beta} + \partial_\nu \hat{g}_{\beta\mu} - \partial_\beta \hat{g}_{\mu\nu} + \partial_\mu h_{\nu\beta} + \partial_\nu h_{\beta\mu} - \partial_\beta h_{\mu\nu}\right)$$

$$= \hat{\Gamma}^{\alpha}{}_{\mu\nu} - h^{\alpha\beta}\hat{g}_{\beta\gamma}\hat{\Gamma}^{\gamma}{}_{\mu\nu} + \frac{1}{2}\hat{g}^{\alpha\beta}\left(\partial_\mu h_{\nu\beta} + \partial_\nu h_{\beta\mu} - \partial_\beta h_{\mu\nu}\right)$$

$$\longrightarrow \boxed{\begin{aligned} \Gamma^{\alpha}{}_{\mu\nu} &= \hat{\Gamma}^{\alpha}{}_{\mu\nu} + \delta\Gamma^{\alpha}{}_{\mu\nu} \\ \delta\Gamma^{\alpha}{}_{\mu\nu} &= \frac{1}{2}\hat{g}^{\alpha\beta}\left(\hat{\nabla}_\mu h_{\nu\beta} + \hat{\nabla}_\nu h_{\beta\mu} - \hat{\nabla}_\beta h_{\mu\nu}\right) \end{aligned}}$$

$\hat{\nabla}_\mu \equiv$ covariant deriv with respect to the background.

Similarly, we find $R^\alpha{}_{\beta\gamma\delta} = \hat{R}^\alpha{}_{\beta\gamma\delta} + \delta R^\alpha{}_{\beta\gamma\delta}$
where $\hat{R}^\alpha{}_{\beta\gamma\delta}$ is assembled only from $\hat{g}_{\alpha\beta}$, and where

$$\delta R^\alpha{}_{\beta\gamma\delta} = \hat{\nabla}_\gamma \delta\Gamma^\alpha{}_{\beta\delta} - \hat{\nabla}_\delta \delta\Gamma^\alpha{}_{\beta\gamma}$$

All the key curvature tensors follow from this, taking the form

$$(\text{tensor}) = (\widehat{\text{tensor}}) + \delta(\text{tensor})$$

usual thing from $\hat{g}_{\alpha\beta}$

Must be treated with care! <u>Not</u> usual tensor made with $h_{\alpha\beta}$.

A few preliminaries before we get to our wave equation:

1. generalized gauge transformations: Again introduce infinitesimal displacement,

$$x^{\alpha'} = x^\alpha + \xi^\alpha, \quad \text{so}$$

$$L^{\alpha'}{}_\beta = \partial_\beta x^{\alpha'} = \delta^\alpha{}_\beta + \partial_\beta \xi^\alpha \rightarrow \text{"small"}$$

Apply to our metric:

$$g_{\mu'\nu'}(x^{\alpha'}) = \hat{g}_{\mu\nu}(x^\alpha + \xi^\alpha) - \hat{g}_{\alpha\nu}\partial_\mu \xi^\alpha - \hat{g}_{\alpha\mu}\partial_\nu \xi^\alpha + h_{\mu\nu}$$

$$= \hat{g}_{\mu\nu}(x^\alpha) + \partial_\alpha \hat{g}_{\mu\nu}\, \xi^\alpha - \hat{g}_{\alpha\nu}\partial_\mu \xi^\alpha - \hat{g}_{\alpha\mu}\partial_\nu \xi^\alpha + h_{\mu\nu}$$

Use $\hat{g}_{\alpha\nu}\partial_\mu \xi^\alpha = \hat{\nabla}_\mu \xi_\nu - \hat{\Gamma}_{\nu\mu\alpha}\xi^\alpha$

$$\rightarrow \boxed{h_{\mu\nu} \rightarrow h_{\mu\nu} - \hat{\nabla}_\mu \xi_\nu - \hat{\nabla}_\nu \xi_\mu}$$

2. Trace reversed perturbation: $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\hat{g}_{\mu\nu} h$

$$h = \hat{g}^{\mu\nu} h_{\mu\nu}$$

Now, assemble Einstein tensor. For simplicity, consider background to be a vacuum solution: $\hat{G}_{\alpha\beta} = \hat{R}_{\alpha\beta} = 0$.

1^{st}, make Ricci. Expand δ(Riemann) out:

$$\begin{aligned}\delta R^{\alpha}{}_{\mu\beta\nu} = \tfrac{1}{2}\Big(&\hat{\nabla}_\beta \hat{\nabla}_\mu h^{\alpha}{}_{\nu} + \hat{\nabla}_\beta \hat{\nabla}_\nu h^{\alpha}{}_{\mu} - \hat{\nabla}_\beta \hat{\nabla}^{\alpha} h_{\mu\nu} \\ &- \hat{\nabla}_\nu \hat{\nabla}_\mu h^{\alpha}{}_{\beta} - \hat{\nabla}_\nu \hat{\nabla}_\beta h^{\alpha}{}_{\mu} + \hat{\nabla}_\nu \hat{\nabla}^{\alpha} h_{\mu\beta}\Big)\end{aligned}$$

$$R_{\mu\nu} = \hat{R}^{\alpha}{}_{\mu\alpha\nu} + \delta R^{\alpha}{}_{\mu\alpha\nu} = \delta R^{\alpha}{}_{\mu\alpha\nu} \quad \text{since } \hat{R}_{\mu\nu} = 0.$$

$$\rightarrow \boxed{\begin{aligned} R_{\mu\nu} &= \tfrac{1}{2}\Big(\hat{\nabla}_\alpha \hat{\nabla}_\mu h^{\alpha}{}_{\nu} + \hat{\nabla}_\alpha \hat{\nabla}_\nu h^{\alpha}{}_{\mu} - \hat{\Box} h_{\mu\nu} \\ &\qquad - \hat{\nabla}_\nu \hat{\nabla}_\mu h - \cancel{\hat{\nabla}_\nu \hat{\nabla}_\alpha h^{\alpha}{}_{\mu}} + \cancel{\hat{\nabla}_\nu \hat{\nabla}^{\alpha} h_{\mu\alpha}}\Big) \\ &= -\tfrac{1}{2}\hat{\Box} h_{\mu\nu} - \tfrac{1}{2}\hat{\nabla}_\nu \hat{\nabla}_\mu h + \hat{\nabla}_\alpha \hat{\nabla}_{(\mu} h_{\nu)}{}^{\alpha} \\ R &= \hat{g}^{\mu\nu} R_{\mu\nu} = -\hat{\Box} h + \hat{\nabla}_\alpha \hat{\nabla}_\beta h^{\beta\alpha} \end{aligned}}$$

SKIP

$$\rightarrow \boxed{\begin{aligned} G_{\mu\nu} = &-\tfrac{1}{2}\hat{\Box} h_{\mu\nu} + \tfrac{1}{2}\hat{g}_{\mu\nu}\hat{\Box} h - \tfrac{1}{2}\hat{\nabla}_\nu \hat{\nabla}_\mu h \\ &+ \hat{\nabla}_\alpha \hat{\nabla}_{(\mu} h_{\nu)}{}^{\alpha} - \tfrac{1}{2}\hat{g}_{\mu\nu}\hat{\nabla}_\alpha \hat{\nabla}_\beta h^{\beta\alpha} \end{aligned}}$$

SKIP

where $\hat{\Box} \equiv \hat{g}^{\mu\nu}\hat{\nabla}_\mu \hat{\nabla}_\nu$ is a covariant

<u>background</u> wave operator.

Switch to trace-reversed metric perturbation:

$$G_{\mu\nu} = -\frac{1}{2}\hat\Box \bar h_{\mu\nu} - \frac{1}{2}\hat g_{\mu\nu}\hat\nabla_\alpha \hat\nabla_\beta \bar h^{\beta\alpha} + \frac{1}{2}\hat\nabla_\alpha \hat\nabla_\mu \bar h_\nu{}^\alpha + \frac{1}{2}\hat\nabla_\alpha\hat\nabla_\nu \bar h_\mu{}^\alpha$$

Interchange derivatives on the second line:

$$\hat\nabla_\alpha \hat\nabla_\mu h_\nu{}^\alpha = \hat\nabla_\mu \hat\nabla_\alpha h_\nu{}^\alpha - \hat R^\beta{}_{\nu\alpha\mu} h_\beta{}^\alpha \cancel{+ \hat R^\alpha{}_{\beta\alpha\mu} h_\nu{}^\beta} \to 0$$

→ Ricci.

$$\rightarrow \boxed{\begin{aligned} G_{\mu\nu} = &-\frac{1}{2}\hat\Box \bar h_{\mu\nu} + \hat R_{\alpha\mu\nu\beta}\bar h^{\alpha\beta} \\ &-\frac{1}{2}\hat g_{\mu\nu}\hat\nabla_\alpha\hat\nabla_\beta \bar h^{\beta\alpha} + \frac{1}{2}\hat\nabla_\mu \hat\nabla_\alpha \bar h^\alpha{}_\nu + \frac{1}{2}\hat\nabla_\nu\hat\nabla_\alpha \bar h^\alpha{}_\mu \end{aligned}}$$

Fix gauge? Would like to set $\hat\nabla_\alpha \bar h^\alpha{}_\mu = \hat\nabla^\alpha \bar h^\alpha{}_\mu = 0$:

$$h^{new}_{\alpha\beta} = h^{old}_{\alpha\beta} - \hat\nabla_\alpha \xi_\beta - \hat\nabla_\beta \xi_\alpha$$

$$\bar h^{new}_{\alpha\beta} = \bar h^{old}_{\alpha\beta} - \hat\nabla_\alpha \xi_\beta - \hat\nabla_\beta \xi_\alpha + \hat g_{\alpha\beta}\hat\nabla^\mu \xi_\mu$$

$$\rightarrow \boxed{\hat\nabla^\alpha \bar h^{new}_{\alpha\beta} = 0 \quad \text{if} \quad \hat\Box \xi_\beta = \hat\nabla^\alpha \bar h^{old}_{\alpha\beta}}$$

"generalized Lorentz gauge"

$$\rightarrow \boxed{G_{\mu\nu} = -\frac{1}{2}\hat\Box \bar h_{\mu\nu} + \hat R_{\alpha\mu\nu\beta}\bar h^{\alpha\beta}}$$

Note: can add function χ_β that satisfies $\hat\Box \chi_\beta = 0$ to gauge generator. Convenient to do this to set

$$(\bar h^{new})^\alpha{}_\alpha = 0 \rightarrow (\bar h^{old})^\alpha{}_\alpha = 2\hat\nabla^\alpha \xi_\alpha$$

"TT gauge". Can now drop bars on h.

Let's look at the gauge fixing more carefully:

$$\bar{h}^{new}_{\alpha\beta} = \bar{h}^{old}_{\alpha\beta} - \hat{\nabla}_\alpha \xi_\beta - \hat{\nabla}_\beta \xi_\alpha + \hat{g}_{\alpha\beta} \hat{\nabla}^\mu \xi_\mu$$

What is the divergence of this?

$$\hat{\nabla}^\alpha \bar{h}^{new}_{\alpha\beta} = \hat{\nabla}^\alpha \bar{h}^{old}_{\alpha\beta} - \hat{\Box} \xi_\beta - \hat{\nabla}^\alpha \hat{\nabla}_\beta \xi_\alpha + \hat{\nabla}_\beta \hat{\nabla}^\mu \xi_\mu$$

Let's focus on that final term:

$$\hat{\nabla}_\beta \hat{\nabla}^\mu \xi_\mu = \hat{\nabla}_\beta \hat{\nabla}_\mu \xi^\mu$$

If we could swap the derivatives, this would cancel with the preceding term. But, when we do this, we pick up a factor of Riemann:

$$\hat{\nabla}_\beta \hat{\nabla}_\mu \xi^\mu = \hat{\nabla}_\mu \hat{\nabla}_\beta \xi^\mu + \hat{R}^\mu{}_{\gamma\mu\beta} \xi^\gamma$$

$$= \hat{\nabla}_\mu \hat{\nabla}_\beta \xi^\mu + \hat{R}_{\gamma\beta} \xi^\gamma$$

$$= \hat{\nabla}_\alpha \hat{\nabla}_\beta \xi^\alpha = \hat{\nabla}^\alpha \hat{\nabla}_\beta \xi_\alpha$$

using vacuum condition & relabeling a dummy index.

$$\rightarrow \hat{\nabla}^\alpha \bar{h}^{new}_{\alpha\beta} = \hat{\nabla}^\alpha \bar{h}^{old}_{\alpha\beta} - \hat{\Box} \xi_\beta$$

$$= 0$$

$$\text{if} \quad \hat{\Box} \xi_\beta = \hat{\nabla}^\alpha \bar{h}^{old}_{\alpha\beta}$$

TOPS

Energy content of waves: Not easy! Can always go into LLF: No "wave" there at all.

Key is <u>NONLOCALITY</u>. Need to examine geometric quantities over some finite spacetime region. Will want to average over a region several wavelengths in size.

Means we need 2^{ND} order theory! 1^{ST} order stuff vanishes when we average.

Put
$$g_{\alpha\beta} = \hat{g}_{\alpha\beta} + \varepsilon h_{\alpha\beta} + \varepsilon^2 j_{\alpha\beta}$$

$$\varepsilon = \text{order counting parameter}$$
$$\equiv 1.$$

Now, examine vacuum Einstein:

$$\begin{aligned} 0 &= G_{\alpha\beta}\left[\hat{g}_{\mu\nu} + \varepsilon h_{\mu\nu} + \varepsilon^2 j_{\mu\nu}\right] \\ &= G_{\alpha\beta}\left[\hat{g}_{\mu\nu}\right] + \\ &\quad \varepsilon G^{(1)}_{\alpha\beta}\left[h_{\mu\nu} ; \hat{g}\right] + \\ &\quad \varepsilon^2 G^{(1)}_{\alpha\beta}\left[j_{\mu\nu} ; \hat{g}\right] + \\ &\quad \varepsilon^2 G^{(2)}_{\alpha\beta}\left[h_{\mu\nu} ; \hat{g}\right] \end{aligned}$$

$G_{\alpha\beta}[\hat{g}] = 0 \rightarrow$ normal background Einstein.

$G^{(1)}{}_{\alpha\beta}[h_{\mu\nu}; \hat{g}] \rightarrow$ 1ST order correction

$G^{(2)}{}_{\alpha\beta}[h_{\mu\nu}; \hat{g}] \rightarrow$ Very messy 2ND order correction to Einstein. Involves lots of terms with $h_{\alpha\beta} \hat{\nabla}_\mu \hat{\nabla}_\nu h_{\rho\sigma}$, $(\hat{\nabla}_\alpha h_{\beta\mu})(\hat{\nabla}_\nu h_{\rho\sigma})$

Require Einstein to hold order by order:

$\mathcal{O}(1)$: $G_{\alpha\beta}[\hat{g}_{\mu\nu}] = 0 \rightarrow$ Background is a vacuum solution.

$\mathcal{O}(\varepsilon)$: $G^{(1)}_{\alpha\beta}[h_{\mu\nu}; \hat{g}] = 0 \rightarrow$ wave equation for $h_{\mu\nu}$.

$\mathcal{O}(\varepsilon^2)$: $G^{(1)}_{\alpha\beta}[j_{\mu\nu}; \hat{g}] = - G^{(2)}_{\alpha\beta}[h_{\mu\nu}; \hat{g}]$

Last terms are quite interesting: The 2ND order perturbation $j_{\mu\nu}$ arises from a source of order h^2.

$\rightarrow \mathcal{O}(h^2)$ tensor acts effectively like a stress energy tensor for the radiation!

Recall separation of lengthscales: $\lambda \ll \mathcal{L}$

Define: $\Delta j_{\mu\nu} = j_{\mu\nu} - \langle j_{\mu\nu} \rangle$

varies on λ

varies on $\mathcal{L}$.

Regroup terms in metric:

$$g_{\mu\nu} = \left[\hat{g}_{\mu\nu} + \varepsilon^2 \langle j_{\mu\nu} \rangle \right] + \left[\varepsilon h_{\mu\nu} + \varepsilon^2 \Delta j_{\mu\nu} \right]$$

vary on $\mathcal{L}$

vary on λ

Average $\mathcal{O}(\varepsilon^2)$ Einstein:

$$\langle G^{(1)}_{\alpha\beta}(j_{\mu\nu}) \rangle = - \langle G^{(2)}_{\alpha\beta}(h_{\mu\nu}) \rangle$$

Useful trick: $\langle \partial^2 f_{\mu\nu} \rangle = \partial^2 \langle f_{\mu\nu} \rangle + \mathcal{O}(\lambda^2/\mathcal{L}^2)$

so

$$G^{(1)}_{\alpha\beta}(\langle j_{\mu\nu} \rangle) = -\langle G^{(2)}_{\alpha\beta}(h_{\mu\nu}) \rangle + \mathcal{O}(\lambda^2/\mathcal{L}^2)$$

or

$$G_{\alpha\beta}\left[\hat{g}_{\mu\nu} + \varepsilon^2 \langle j_{\mu\nu} \rangle \right] = - \langle G^{(2)}_{\alpha\beta}(h_{\mu\nu}) \rangle$$

This term acts as an Einstein source for all "long" metric degrees of freedom!

Suggests a definition:

$$T^{GW}_{\alpha\beta} = \frac{-1}{8\pi G} \langle G^{(2)}_{\alpha\beta}(h_{\mu\nu}) \rangle$$

Straightforward but tedious calculation:

$$T^{GW}_{\alpha\beta} = \frac{1}{32\pi G}\Big\langle \hat{\nabla}_\alpha \bar{h}_{\mu\nu} \hat{\nabla}_\beta \bar{h}^{\mu\nu} - \frac{1}{2}\hat{\nabla}_\alpha \bar{h}\,\hat{\nabla}_\beta h$$
$$- \hat{\nabla}_\alpha \bar{h}_{\beta\gamma} \hat{\nabla}_\mu \bar{h}^{\mu\gamma} - \hat{\nabla}_\beta \bar{h}_{\alpha\gamma} \hat{\nabla}_\mu \bar{h}^{\mu\gamma}\Big\rangle$$

Choose gauge to kill divergences, kill trace:

$$\boxed{T^{GW}_{\alpha\beta} = \frac{1}{32\pi G}\left\langle \hat{\nabla}_\alpha h^{TT}_{\mu\nu} \hat{\nabla}_\beta h^{TT\,\mu\nu}\right\rangle}$$

"Isaacson stress-energy tensor"

Common application: energy flux in a nearly flat region.

$$T_{00} = \frac{dE}{dA\,dt} = \frac{1}{32\pi G}\left\langle \partial_t h^{TT}_{ij}\, \partial_t h^{TT}_{ij}\right\rangle$$

$\hookrightarrow$ nearly flat: $\nabla_t \to \partial_t$

$$\frac{dE}{dt} = r^2 \int d\Omega\, T_{00}$$

Put $h^{TT}_{ij} = \frac{2G}{r}\ddot{I}_{kl}\left[P_{ki}P_{lj} - \frac{1}{2}P_{kl}P_{ij}\right]$

$$P_{ij} = \delta_{ij} - n_i n_j$$

$$\int n_i n_j\, d\Omega = \frac{4\pi}{3}\delta_{ij} \qquad \int n_i n_j n_k\, d\Omega = 0$$

$$\int n_i n_j n_k n_l\, d\Omega = \frac{4\pi}{15}\left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$$

$$\longrightarrow \quad \frac{dE}{dt} = \frac{G}{5}\left\langle \dddot{I}_{ij}\,\dddot{I}_{ij}\right\rangle$$

"Quadrupole formula"