Cosmology: Study of the large scale structure of the universe.

1st example of spacetime constructed by symmetry argument! No weak field assumption: Deduce form of metric by restrictions of symmetry. Apply Einstein: symmetric form reduces nonlinear nastiness to something OK.

Background: "Maximally symmetric spaces". Carroll Sec 3.9 gives extensive discussion.

A "maximally symmetric space" has the largest number of possible Killing vectors: $n(n+1)/2$ (for n-D space). Intuitively, it is <u>homogeneous</u> (uniform properties in all locations) and <u>isotropic</u> (looks the same in all directions). (Isotropy $\leftrightarrow$ invariant wrt rotations w/ boosts; homogeneous $\leftrightarrow$ invariant wrt translations.)

Example: Flat spacetime: 3 boosts, 3 rotations, 4 translations. $(n=4)$

Euclidean space: 3 rotations, 3 translations $(n=3)$

These requirements lead to condition that Riemann must be Lorentz invariant within LLF: can be no unique direction.

Doing this and recovering symmetries of Riemann leads to

$$R_{\alpha\mu\beta\nu} = \frac{R}{n(n-1)}\left(g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\beta\mu}\right)$$

in n-D. R is a <u>constant</u>.

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Ricci: $$R_{\mu\nu} = \frac{R}{n(n-1)}\left(n g_{\mu\nu} - g_{\mu\nu}\right) = \frac{R g_{\mu\nu}}{n}$$

$$g^{\mu\nu} R_{\mu\nu} \equiv R. \quad \text{Duh.}$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = R g_{\mu\nu}\left(\frac{1}{n} - \frac{1}{2}\right)$$

Notice: maximally symmetric spacetimes are those in which the only stress energy tensor is a cosmological constant! (or vacuum)

The universe: Begin with observation that on large scales the universe is homogeneous and isotropic.

Only true SPATIALLY: Past is very different from present! Not temporally isotropic, but spatial distribution is homogeneous, and no evidence for anisotropy has turned up.

What is "Large enough"? Largest scales observed on CMB: homogeneous to 10^{-5}. No evidence for deviations from homogeneity until we get down to scales $\sim$ several $\times$ 10 Mpc - galaxy cluster scales. Small scale inhomogeneity due to gravitational clumping.

We take spacetime metric to have the form

$$ds^2 = -dt^2 + R^2(t)\,\gamma_{ij}\,dx^i dx^j$$

$R(t) \equiv$ "scale factor", NOT Ricci scalar. Unfortunate notation ... oh well.

Set $g_{tt} = -1$, $g_{ti} = 0$: "Co-moving coordinates". An observer "at rest" in these coords, $\vec{u} \doteq (1,0,0,0)$ is just "comoving" with the spacetime. Note: Earth is NOT comoving! Leads to an observed anisotropy (dipole of CMB). Leading contributor is infall of Milky Way into the Virgo cluster.

Choose metric γ_{ij} to be maximally symmetric. Coordinates x^i are dimensionless - all lengthscales are in $R(t)$. Controls whether space expands or contracts as universe evolves.

Riemann we build from the spatial metric γ_{ij} is

$${}^{(3)}R_{ijkl} = k\left(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}\right)$$

$${}^{(3)}R_{jl} = 2k\,\gamma_{jl}$$

$6k$ = Ricci scalar; more convenient to just leave it as k for now.

Since γ_{ij} is isotropic, it must be spherically symmetric:

$$dx^i dx^j \gamma_{ij} = f(\bar{r})\,d\bar{r}^2 + \bar{r}^2 d\Omega^2 \qquad \bar{r} = \text{dimensionless radial coord.}$$

$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$$

Convenient to put $f(\bar{r}) = \exp(2\beta(\bar{r}))$; gives us a very handy result for ${}^{(3)}$Ricci built from γ_{ij}:

$$ {}^{(3)}R_{\bar{r}\bar{r}} = \underbrace{\frac{2}{\bar{r}}\,\partial_{\bar{r}}\beta}_{\leftarrow \text{ from math}} = 2k\,\gamma_{\bar{r}\bar{r}} \quad \leftarrow \text{from symmetric form.} $$

$$ \rightarrow \qquad \frac{\partial_{\bar{r}}\beta}{\bar{r}} = k e^{2\beta} $$

$$ e^{-2\beta}\,\partial_{\bar{r}}\beta = k\bar{r} \qquad \beta = 0 \;@\; \bar{r} = 0 $$

$$ \longrightarrow \quad \beta = -\frac{1}{2}\ln\left[1 - k\bar{r}^2\right] $$

So our line element becomes

$$ ds^2 = -dt^2 + R^2(t)\left[\frac{d\bar{r}^2}{1 - k\bar{r}^2} + \bar{r}^2 d\Omega^2\right] $$

If we put $k' = \alpha k$, but define $\tilde{r} = \sqrt{\alpha}\,\bar{r}$, $\tilde{R} = (1/\sqrt{\alpha})R$, we have

$$ ds^2 = -dt^2 + \tilde{R}^2(t)\left[\frac{d\tilde{r}^2}{1 - k'\tilde{r}^2} + \tilde{r}^2 d\Omega^2\right] $$

→ Normalization can be absorbed into $R(t)$, just put $k = \{-1, 0, 1\}$.

(Drop prime on k, tilde on r.)

Common notation: Define a radial coordinate χ via

$$d\chi = \frac{d\bar{r}}{\sqrt{1-k\bar{r}^2}} \longrightarrow \begin{array}{ll} \bar{r} = \sin\chi & k=+1 \\ = \chi & k=0 \\ = \sinh\chi & k=-1 \end{array}$$

So

$$ds^2 = -dt^2 + R^2(t)\left[d\chi^2 + \sin^2\chi\, d\Omega^2\right] \quad k=+1$$

→ Each spatial slice is a 3-sphere. "Closed".

$$ds^2 = -dt^2 + R^2(t)\left[d\chi^2 + \chi^2\, d\Omega^2\right] \quad k=0$$

→ Each spatial slice is Euclidean. "Flat"

$$ds^2 = -dt^2 + R^2(t)\left[d\chi^2 + \sinh^2\chi\, d\Omega^2\right] \quad k=-1$$

→ Each spatial slice is a hyperboloid. "Open".

Carroll introduces another notation: Choose a particular value of the scale factor: $R(t) = R_0$. Define

$$a(t) = R(t)/R_0, \quad r = R_0\bar{r}, \quad \kappa = k/R_0^2$$

Then,

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-\kappa r^2} + r^2 d\Omega^2\right]$$

More common notation: $R(t)$ in previous stuff is denoted $a(t)$.

Useful choice: $R_0 = R(\text{now}) \rightarrow a(\text{now}) = 1$.

"Robertson-Walker" metrics.

Note: $a=0 \rightarrow$ spatial slice has zero volume! Universe with this as an initial condition + then expands goes through a "Big Bang".

So far, just discussing geometry - no contact with the stuff which acts as source yet.

Chose our source to be a perfect fluid; to satisfy the isotropy / homogeneity requirement, this fluid must be at rest in comoving coordinates. Then,

$$T_{\mu\nu} = (\rho + P) u_\mu u_\nu + P g_{\mu\nu} \doteq \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \gamma_{ij} P & \\ 0 & & & \end{pmatrix}$$

Handily, $T^\mu{}_\nu \doteq \mathrm{diag}(-\rho, P, P, P)$

An important relation is given by

$$\nabla_\mu T^\mu{}_0 = 0 \rightarrow \text{local conservation of energy.}$$

$$\rightarrow \quad 0 = \partial_\mu T^\mu{}_0 + \Gamma^\mu_{\mu\lambda} T^\lambda{}_0 - \Gamma^\lambda_{\mu 0} T^\mu{}_\lambda$$

$$\rightarrow \quad \boxed{0 = \partial_t \rho + 3 \frac{\partial_t a}{a} (\rho + P)}$$

An equivalent form of this:

$$\partial_t (\rho R^3) = - P \, \partial_t (R^3)$$

Rate of change of energy in a volume

work done by fluid as it expands.

Ie: $\quad du = - P dV.$

Now apply Einstein field equation. Write this in the form

$$R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} T \right)$$

R_{00} : $$\left(\frac{\ddot{a}}{a}\right) = -\frac{4\pi G}{3}\left(\rho + 3P\right) \qquad (F2)$$

$R_{ii} + R_{00}$: $$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2} \qquad (F1)$$

These are the Friedmann equations.

FRW - "Friedmann-Robertson-Walker" - spacetimes are RW with scale factor governed by Friedmann.

Terminology: $H \equiv \dot{a}/a \rightarrow$ The Hubble expansion parameter.

$H_0 \equiv$ Hubble "constant" - value of H today.

Best data: $H_0 = 73 \pm 3$ (km/sec) Mpc^{-1}

Notice $[H] \sim (\text{time})^{-1}$.

$$\Omega = \frac{\rho}{\rho_{crit}} \quad \text{where} \quad \rho_{crit} = \frac{3H^2}{8\pi G} = \text{"critical density"}$$

This definition allows us to write F1 as

$$\Omega - 1 = \frac{K}{H^2 a^2}$$

Can now see how density of universe affects its overall spatial curvature:

$\Omega < 1 \rightarrow K < 0$ open universe

$\Omega = 1 \rightarrow K = 0$ flat

$\Omega > 1 \rightarrow K > 0$ closed.

Common notational trickery. Define

$$\rho_c \equiv -\frac{3K}{8\pi G a^2}, \quad \Omega_c \equiv \frac{\rho_c}{\rho_{crit}} \equiv \frac{-K}{H^2 a^2} \rightarrow \Omega + \Omega_c = 1$$

"curvature density" → not really a density!

Further progress: Choose an equation of state. Most common parameterization used in cosmology is

$$P = w\rho$$

where w = constant. Restrictive! — NOT the form we will use to discuss stars. Useful to describe matter in cosmology.

For intuition, consider universes dominated by a single species of stuff: $\rho = \rho_i$ to high approximation → $P = w_i \rho_i$

Run ~~this~~ this model through ~~the~~ stress energy tensor conservation: $\partial_t(\rho a^3) = -P\partial_t(a^3)$

$$\longrightarrow \frac{\dot\rho}{\rho} = -3(1+w)\frac{\dot a}{a}$$

$$\longrightarrow \frac{\rho}{\rho_0} = \left(\frac{a}{a_0}\right)^{-3(1+w)}$$ Put $a_0 = 1$.

Examples of ~~matter~~ "stuff":

1. Matter: By convention, "matter" has $P=0$ → $w=0$.
   Ie, matter is dust! Seems stupid at 1st blush...
   but typical "dust particle" is a galaxy — "matter" really is pressureless on that scale. Evolution

   $\rho_M \propto a^{-3}$ → number stays fixed, volume grows as a^3.

2. Radiation: From stat mech, $P_R = \frac{1}{3}\rho_R \rightarrow w = 1/3$.

Evolution: $\rho_R \propto a^{-4}$. Volume goes as a^3; each quantum loses energy as $a^{-1} \rightarrow$ redshift!

3. Cosmological constant: $P_\Lambda = -\rho_\Lambda \rightarrow w = -1$.

Evolution: $\rho_\Lambda \propto a^0 \rightarrow$ constant!

Vacuum energy independent of universe's scale.

Now, examine a mono-species universe. For simplicity, take a flat universe $(k=0)$, examine Friedmann:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G \rho}{3} = \frac{8\pi G \rho_0 a^{-n}}{3} \qquad n = 3(1+w)$$

$\rightarrow \quad \dot{a} \propto a^{1-n/2}$ (Positive root: expansion!)

$\rightarrow \quad \boxed{a \propto t^{2/n} \quad \text{except for } n=0}$

Matter dominated: $w=0 \rightarrow n=3$

$\rightarrow \quad a \propto t^{2/3}$

Radiation dominated: $w = 1/3 \rightarrow n = 4$

$\rightarrow \quad a \propto t^{1/2}$

Cosmological constant: $n=0$.

Treat separately ...

$$\rho = \rho_0 = \frac{\Lambda}{8\pi G}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\Lambda}{3}$$

$$\rightarrow \quad a \propto \exp\left[\pm\sqrt{\frac{\Lambda}{3}}\, t\right]$$

Exponential expansion.

Real universe not so simple - have a mixture of these.

But, examples are illustrative.

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