Spacetime of a <u>compact</u> spherical body: compact means that the body occupies a finite spatial region; spacetime is vacuum in the exterior.

Most general static form of the metric:

$$ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + R^2(r)\,d\Omega^2$$

Key thing for sphericity: All functions only depend on r, and angular sector has the form of a 2-sphere.

⌈ Why no cross term? Isn't

$$ds^2 = -a^2(dt')^2 - 2ab\,dt'\,dr + c^2dr^2 + R^2d\Omega^2$$

more general?

No: Put $e^{\Phi}dt = a\,dt' + b\,dr$. If we define $e^{2\Lambda} = b^2 + c^2$, recover original form. ⌋

Removal of cross term amounts to a clever choice of time coordinate. Can similarly make a clever choice of radial coordinate: Put $R(r) \equiv r$. This means r is an <u>AREAL</u> radius - it labels spheres of area $4\pi r^2$.

Then, we have

$$ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + r^2d\Omega^2$$

"Schwarzschild coordinates"

Before moving on. useful to know about other coordinates.
Particularly good are isotropic coordinates:

$$ds^2 = -e^{2\Phi(\bar{r})}\,dt^2 + e^{2\mu(\bar{r})}\left[d\bar{r}^2 + \bar{r}^2 d\Omega^2\right]$$

These coordinates emphasize the isotropy of the 3 spatial directions ... but, lose areal interpretation of r.

Weak field: recall line element

$$ds^2 = -(1+2\Phi)\,dt^2 + (1-2\Phi)\left[dx^2+dy^2+dz^2\right]$$

→ Isotropic with $\mu = -\Phi$, $|\mu| \ll 1$, $|\Phi| \ll 1$.

$\Phi = -\dfrac{GM}{\bar{r}}$ in ~~interior~~ exterior

$=$ solution of $\nabla^2\Phi = -4\pi G\rho$ in interior.

We will expect to construct a similar solution for our general metric, with different solutions for "exterior" and "interior" that must be matched at surface.

Switch back to Schwarzschild coordinates...

What we need: 1. Curvature tensors; 2. Matter terms

Curvature is straightforward: See Carroll p 195 but replace $\alpha \to \Phi$, $\beta \to \Lambda$. Key pieces:

2ND $\longrightarrow$ $$R_{tt} = e^{2[\Phi - \Lambda]}\left[\partial_r^2 \Phi + (\partial_r \Phi)^2 - (\partial_r \Phi)(\partial_r \Lambda) + \frac{2\partial_r \Phi}{r}\right]$$

1ST $\longrightarrow$ $$R_{rr} = -\left[\partial_r^2 \Phi + (\partial_r \Phi)^2 - (\partial_r \Phi)(\partial_r \Lambda) - \frac{2\partial_r \Lambda}{r}\right]$$

$$R_{\theta\theta} = e^{-2\Lambda}\left[r\partial_r \Lambda - r\partial_r \Phi - 1\right] + 1$$

$$R_{\phi\phi} = \sin^2\theta \, R_{\theta\theta}$$

From this, simple to build Einstein tensor. Also need to know source. Consider static perfect fluid, and so put

$$T_{\mu\nu} = (\rho + P) u_\mu u_\nu + P g_{\mu\nu}$$

$\rho = \rho(r)$ energy density in fluid rest frame.

$P = P(r)$ pressure in fluid rest frame.

Static condition (hydrostatic equilibrium) forces us to put

$$\vec{u} \doteq (u^t, 0, 0, 0)$$

$$\vec{u}\cdot\vec{u} = -1 \rightarrow u^t = e^{-\Phi}, \quad u_t = -e^{\Phi}.$$

Finally assume $\rho(r) = 0$, $P(r) = 0$ for $r \geq R_*$, the surface of the "star".

Begin by considering exterior: $T_{\mu\nu}=0$. Simple!

$$G_{\mu\nu}=0 \rightarrow R_{\mu\nu}=0$$

All components vanish ... we can assemble some convenient combinations of components.

$$e^{2[\Lambda-\Phi]} R_{tt} + R_{rr} = 0$$

$$\rightarrow \frac{2\partial_r\Phi}{r} + \frac{2\partial_r\Lambda}{r} = 0 \longrightarrow \Phi = -\Lambda + k$$

k is an integration constant. Plug into metric, see that it's just a coordinate rescaling: $t \rightarrow e^{k}t$. So, set it to zero.

Next, examine $R_{\theta\theta}$:

$$e^{-2\Lambda}\left[r\partial_r\Lambda - r\partial_r\Phi - 1\right] + 1 = 0$$

$$e^{2\Phi}\left[2r\partial_r\Phi + 1\right] = 1$$

$$\rightarrow \partial_r\left[r e^{2\Phi}\right] = 1$$

$$\rightarrow \Phi = \frac{1}{2}\ln\left[1 + \frac{A}{r}\right]$$

Fix A later. For now, line element is

$$ds^2 = -\left(1 + \frac{A}{r}\right)dt^2 + \left(1 + \frac{A}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

To fix A, consider non-relativistic motion in weak field ($r \gg A$). Also, consider purely radial free fall:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$$

$$\rightarrow \frac{d^2r}{d\tau^2} + \Gamma^r_{00}\left(\frac{dt}{d\tau}\right)^2 = 0 \qquad \left(\frac{dt}{d\tau} \gg \frac{dr}{d\tau}\right)$$

↳ nonrel limit

$$\Gamma^r_{00} = e^{2(\Phi-\Lambda)}\partial_r\Phi$$

$$= e^{4\Phi}\partial_r\Phi = -\frac{A}{2r^2}\left(1+\frac{A}{r}\right) \simeq \frac{-A}{2r^2}$$

$$\rightarrow \frac{d^2r}{dt^2} = \frac{A}{2r^2}$$

Newtonian free fall: $\frac{d^2r}{dt^2} = -\frac{GM}{r^2} \longrightarrow \boxed{A = -2GM}$

$$\boxed{ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1}dr^2 + r^2 d\Omega^2}$$

"The Schwarzschild metric"

Significance due to Birkhoff's Theorem:

The exterior, vacuum region of ANY spherically symmetric body is described by the Schwarzschild metric.

True even if the source is TIME VARYING as long as the variations preserve spherical symmetry (e.g., radial pulsations).

To see this, write the general line element with time dependence:

$$ds^2 = -e^{2\Phi(r,t)}dt^2 + e^{2\Lambda(r,t)}dr^2 + r^2 d\Omega^2$$

Now, work out curvature and enforce $G_{\mu\nu} = 0 \rightarrow R_{\mu\nu} = 0$. (See Carroll p 202) New component enters:

$$R_{tr} = \frac{2}{r}\partial_t \Lambda = 0 \rightarrow \Lambda = \Lambda(r)$$

Other components are similar, but generally pick up some new time derivative terms. $R_{\theta\theta}$ turns out to be unchanged:

$$R_{\theta\theta} = e^{-2\Lambda}\left[r\partial_r\Lambda - r\partial_r\Phi - 1\right] + 1 = 0$$

Hence $\partial_t R_{\theta\theta} = 0 \longrightarrow \partial_t\partial_r\Phi = 0$

$$\rightarrow \Phi = \Phi_r(r) + \Phi_t(t)$$

Our most general spherical spacetime thus appears to be

$$ds^2 = -e^{2\Phi_r(r)} e^{2\Phi_t(t)} dt^2 + e^{2\Lambda(r)} dr^2 + r^2 d\Omega^2$$

Now, we can simplify by changing the time coordinate:

$$dt \rightarrow e^{\Phi_t(t)} dt$$

Amounts to choosing time coordinate such that $\Phi(r,t) = \Phi(r)$.

Note this theorem demonstrates that any spherically symmetric vacuum spacetime has a timelike Killing vector: $\partial_t g_{\alpha\beta} = 0$ $\rightarrow \vec{\xi}^t$ exists. Means we can define a conserved "energy" $E = -\vec{p} \cdot \vec{\xi}^t$, which will be useful for orbits.

How about interior? Repeat exercise using perfect fluid $T_{\mu\nu}$.

Most convenient form is $G^\mu{}_\nu = 8\pi G T^\mu{}_\nu$, since

$$T^\mu{}_\nu = \mathrm{diag}\left(-\rho(r), P(r), P(r), P(r)\right).$$

Previous $R_{\mu\nu}$ gives us $G^\mu{}_\nu$.

2 interesting components:

$$G^t{}_t = -\frac{1}{r^2}\frac{d}{dr}\left[r\left(1-e^{-2\Lambda}\right)\right]$$

Useful to make a definition: $e^{-2\Lambda} = 1 - \frac{2Gm(r)}{r}$ ← nicely hooks up to exterior.

$$\rightarrow \quad G^t{}_t = -\frac{2G}{r^2}\frac{dm(r)}{dr} = 8\pi G T^t{}_t$$

$$= -8\pi G \rho(r)$$

$$\rightarrow \quad m(r) = \int_0^r 4\pi\rho(r')(r')^2\,dr'$$

Note: B.C. that $m(0) = 0$. If not true, get a black hole!

This $m(r)$ defines the mass of the star. Note this is NOT the volume integral of the density - missing a factor $\sqrt{g_{rr}}$ needed to make a proper volume integral.

$$m(r) < \int_0^r 4\pi\rho(r')(r')^2 e^{\Lambda(r')}\,dr'.$$

The "missing" mass/energy can be regarded as gravitational binding energy.

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$$G^r{}_r = e^{-2\Lambda}\left[\frac{2}{r}\frac{d\Phi}{dr} + \frac{1}{r^2}\right] - \frac{1}{r^2}$$

$$= \left(1 - \frac{2Gm}{r}\right)\left[\frac{2}{r}\frac{d\Phi}{dr} + \frac{1}{r^2}\right] - \frac{1}{r^2}$$

$$= 8\pi G T^r{}_r = 8\pi G P$$

$$\rightarrow \boxed{\frac{d\Phi}{dr} = \frac{G\left(m + 4\pi r^3 P\right)}{r\left(r - 2Gm\right)}}$$

Old homework exercise: Showed that $\nabla_\mu T^{\mu\nu} = 0$ for perfect fluid in hydrostatic equilibrium implies

$$(\rho + P)\, u^\beta \nabla_\beta u_\alpha = -\partial_\alpha P - u_\alpha u^\beta \partial_\beta P$$

Apply to this spacetime $\left(\vec{u} \doteq (e^{-\Phi}, 0, 0, 0)\right)$:

$$(\rho + P)\,\Gamma^t{}_{rt} = (\rho + P)\frac{d\Phi}{dr} = -\frac{dP}{dr}$$

$$\rightarrow \boxed{\frac{dP}{dr} = -\frac{G(\rho + P)(m + 4\pi r^3 P)}{r(r - 2Gm)}}$$

Equations for $m(r)$, $d\Phi/dr$, dP/dr are the Tolman-Oppenheimer-Volkov equations of stellar structure.

To solve them, we need 2 things:

1. An equation of state that relates P & ρ:
2. An initial condition (ρ_c or P_c)

Specify: then "integrate up" to build a "star" in GR!