Last time: Determined that the spacetime of a spherically symmetric body is given by

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1-2Gm(r)/r} + r^2 d\Omega^2$$

using "Schwarzschild coordinates", and where the body has

$$P = P(r) \qquad r \le R_*$$
$$\rho = \rho(r)$$
$$\rho = P = 0 \qquad r > R_*$$

Enforcing vacuum Einstein (appropriate for $r > R_*$) leads to

$$m(r) = m(R_*) \equiv M_{TOT}$$
$$e^{2\Phi(r)} = 1 - 2GM_{TOT}/r$$

Enforcing Einstein in the interior leads to

$$m(r) = 4\pi\int_0^r \rho(r')(r')^2\,dr'$$

$$\frac{d\Phi}{dr} = \frac{G(m(r) + 4\pi r^3 P(r))}{r(r-2Gm(r))}$$

Enforcing $\nabla_\mu T^\mu{}_\nu = 0$ leads to

$$\frac{dP}{dr} = -\frac{G(\rho+P)(m+4\pi r^3 P)}{r(r-2Gm)}$$

"TOV equations". Non-rel, Newtonian limit:

$$\frac{d\Phi}{dr} = \frac{Gm(r)}{r^2}$$

$$\frac{dP}{dr} = -\frac{G\rho(r)m(r)}{r^2}$$

Highly idealized, unrealistic - but instructive! - limit: $\rho = \text{constant}$.
"Star" with sound speed, $c_s^2 = dP/d\rho = \infty$! Not physical.

Trivial mass function:

$$m(r) = \frac{4}{3}\pi \rho r^3 \qquad r \leq R_*$$

$$= \frac{4}{3}\pi \rho R_*^3 \qquad r > R_*$$

Pressure profile much more complicated:

$$\frac{dP}{dr} = -\frac{4}{3}\pi r G \left[\frac{(\rho + P)(\rho + 3P)}{1 - 8\pi G \rho r^2/3}\right]$$

Miracle: This has a simple solution.

$$\left[\frac{\rho + 3P}{\rho + P}\right] = \left[\frac{\rho + 3P_c}{\rho + P_c}\right]\left(1 - \frac{2Gm(r)}{r}\right)^{1/2}$$

where $P_c = P[r=0]$ - pressure at center.

Using the fact that $P \to 0$ at $r = R_*$, can relate stellar radius to choice of central pressure:

$$R_*^2 = \frac{3}{8\pi G \rho}\left[1 - \frac{(\rho + P_c)^2}{(\rho + 3P_c)^2}\right]$$

-or-

$$P_c = \frac{\rho\left[1 - (1 - 2GM_{TOT}/R_*)^{1/2}\right]}{3\sqrt{1 - \frac{2GM_{TOT}}{R_*}} - 1}$$

Have a one parameter family of models:

Pick P_c, get R_* ... or vice versa.

Formula for P_c diverges for a certain "compactness":

$$P_c \to \infty \quad \text{as} \quad 3\sqrt{1 - \frac{2GM_{TOT}}{R_*}} - 1 = 0$$

$$1 - \frac{2GM_{TOT}}{R_*} = \frac{1}{9}$$

$$\longrightarrow \boxed{\frac{GM_{TOT}}{R_*} = \frac{4}{9}}$$

Solutions imply a "maximum compactness": For uniform density stars, we cannot have physically reasonable pressure profiles if $\frac{R_*}{GM} < \frac{9}{4}$.

This limit holds in general - known as Buchdahl's theorem: No stable, spherical fluid configuration can exist if $R_* < 9GM_*/4$. Only requirement: Need $dP/dr < 0$ over star. See Weinberg, Sec 11.6.

What if such a star existed? It would not be stable! Need to consider time evolution... to be discussed soon.

Real stars: Not constant density!

general case: $P = P(\rho)$

Even more general case: $P = P(\rho, S)$ where S = entropy.

For applications in which general relativity is important - e.g., structure of neutron stars - the fluid is often so cold that entropy decouples: via $dU = -P\,dV + T\,dS$, can ignore entropy.

What does cold mean? Depends on situation! For neutron stars, we are dealing with a Fermi fluid, so relevant scale is set by Fermi temperature:

$$T_F = \frac{E_F}{k_B} = \frac{\sqrt{p_F^2 + m^2 c^4}}{k_B}$$

$$p_F = \left[\frac{\rho h^3}{8\pi m}\right]^{1/3}$$

Plug in numbers appropriate for a neutron star: $\rho \sim 10^{16}\ \mathrm{gm/cm^3}$, $m = m_N \rightarrow T_F \sim 10^{13}\ \mathrm{K}$.

Observations: $T \sim 10^6 - 10^9\ \mathrm{K} \ll T_F$

↳ supernova!

So - cold! At least for this example.

An approximation, useful for test cases: Pressure is a power law as a function of density:

$$P = K \rho_0^{\Gamma} \qquad K, \Gamma \text{ constants.}$$

"Polytrope"

(Real form: Typically much more complicated, but often well-described as piecewise polytropic.)

CAUTION: Subtlety here! The density used here is "rest mass density", ρ_0. Does not take into account increase in energy density due to work by pressure squeezing the fluid.

To account for this, invoke 1^{ST} law of thermodynamics:

$$dU = -P\,dV \longrightarrow \text{Work done on fluid element.}$$

Total energy in fluid element

Now, use $\rho = (U/\text{Volume})$, $\rho_0 = (m_{rest}/\text{Volume})$ to rewrite the 1^{ST} law as

$$d\left(\frac{\rho}{\rho_0}\right) = -P\,d\left(\frac{1}{\rho_0}\right) \qquad (1^{ST} \text{ law per unit rest frame})$$

$$= \frac{K^{1/\Gamma}}{\Gamma} \frac{P\,dP}{P^{1+1/\Gamma}} \qquad \leftarrow \text{only true for polytrope}$$

$$\longrightarrow \quad \rho = \frac{P}{\Gamma - 1} + \text{constant}$$

Constant: $\rho \to \rho_0$ as $P = 0$: constant $= \rho_0$.

$$\rightarrow \quad \boxed{\rho = \rho_0 + \frac{P}{\Gamma - 1} = \rho_0 + \frac{K \rho_0^{\Gamma}}{\Gamma - 1}}$$

Recipe to build a stellar model:

1. Pick $\rho_0 (r=0)$ → implies ρ_c, P_c

2. Set $m(r) = 0$ at $r=0$.

3. Integrate dP/dr, $m(r)$ from $r=0$ using a numerical integrator. Optional: Also integrate $d\Phi/dr$ from center. It's initial value is not known - set it to zero for now.

4. When you reach $P=0$, you've hit the surface of the star: $P(r)=0$ defines $r = R_*$. Now know the mass + radius.

Boundary condition: By Birkhoff's theorem, we know $\Phi = \frac{1}{2} \ln(1-2GM/r)$ for $r > R_*$, so

$$\Phi(R_*) = \frac{1}{2} \ln\left(1 - 2GM/R_*\right).$$

Now adjust solution for Φ to match this boundary condition.

Consider spacetime that is Schwarzschild

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

for <u>all</u> r, not just the exterior of some object.

This is an exact, <u>VACUUM</u> solution - but it has orbits / geodesics indicating it has a mass M!

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu{}_{\alpha\beta} u^\alpha u^\beta = 0 \xrightarrow[\text{field}]{\text{weak}} \frac{d^2 x^i}{d\tau^2} + \Gamma^i{}_{00} u^0 u^0 = 0$$

$$\rightarrow \quad \frac{d^2 x^i}{dt^2} = \frac{\partial}{\partial x^i}\left(\frac{GM}{r}\right)$$

What is vacuum ... but has a mass M?

Analogous to Coulomb point charge:

$$\underset{\sim}{E} = \frac{q \underset{\sim}{x}}{r^3} \rightarrow \underset{\sim}{\nabla} \cdot \underset{\sim}{E} = 4\pi\rho = 0$$

No density ... but total charge must be q.

Resolution: Singular point charge at $r=0$.

Expect similar resolution for our case: Schwarzschild solves $G_{\mu\nu} = 0$ everywhere ... but $r=0$ might be a bit odd.

(Even more odd than in Maxwell case thanks to nonlinearities!)

Now look at spacetime itself. Two radii look troublesome:

$r = 2GM$, $r = 0$.

Metric components can be deceiving, so examine curvature (Caroll, Eq 5.13 gives non-zero Riemann components).

From this, an invariant curvature measure:

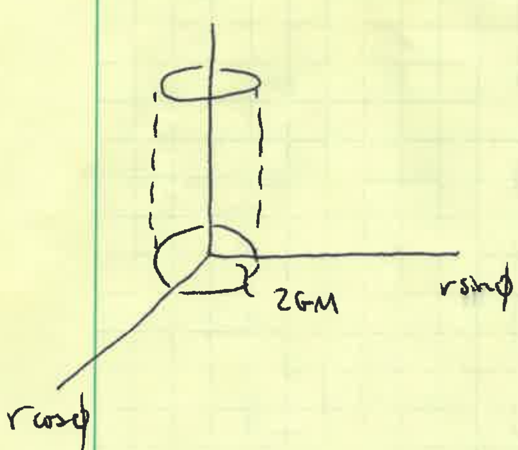
$$I \equiv R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} \equiv \text{"Kretschmann scalar"}$$
$$= \frac{48 G^2 M^2}{r^6}$$

$\sqrt{I}$ is roughly an invariant characterization of tidal forces felt by a body in the spacetime.

Notice: <u>NOT</u> singular at $r = 2GM$, but singular at $r = 0$.

→ Reminiscent of Coulomb point charge!

So, what is $r = 2GM$? Consider the circle $r = 2GM$, $\theta = \pi/2$ as time advances: what is the spacetime area of the tube this circle sweeps out?



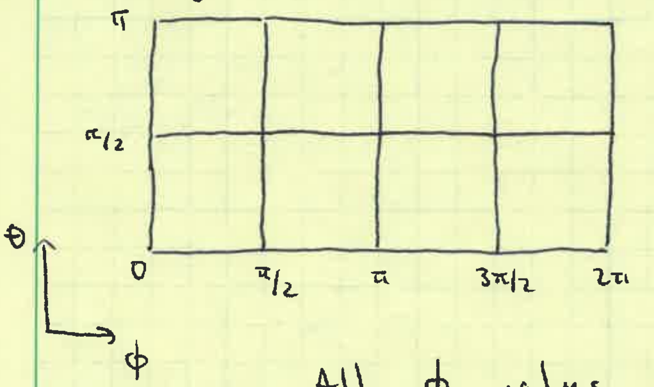
$$A_{tube} = \int_{t_{start}}^{t_{end}} dt \int_0^{2\pi} d\phi \times \left[g_{tt}\, g_{\phi\phi} \right]^{1/2}_{r=2GM,\ \theta=\pi/2}$$

$$= 2GM \int dt \int_0^{2\pi} d\phi \left[1 - \frac{2GM}{r} \right]^{1/2}_{r=2GM}$$

$$= 0$$

"World tube" has no surface area!

Issue is that coordinate t is pathological there! Consider analogy: Draw a sphere as follows:



Perfectly accurate rendering of a sphere's coordinates ... but a horrible rendering of its geometry.

All ϕ values at $\theta = 0$, $\theta = \pi$ occupy a single point. This drawing tempts us to compute quantities like "length" of the $\theta = 0$ slice of the sphere.

Tube drawing likewise represents Schwarzschild coordinates, but distorts the geometry. In fact, all times t map to a single sphere at $r = 2GM$!

Insight into what happens near that radius requires us to select more appropriate time coordinate.

Example: Drop a particle from rest at $r = r_0$, integrate geodesic equation to find its motion as a function of "time"

"time" $\equiv$ coordinate time t &

proper time as measured by the infalling body, τ.

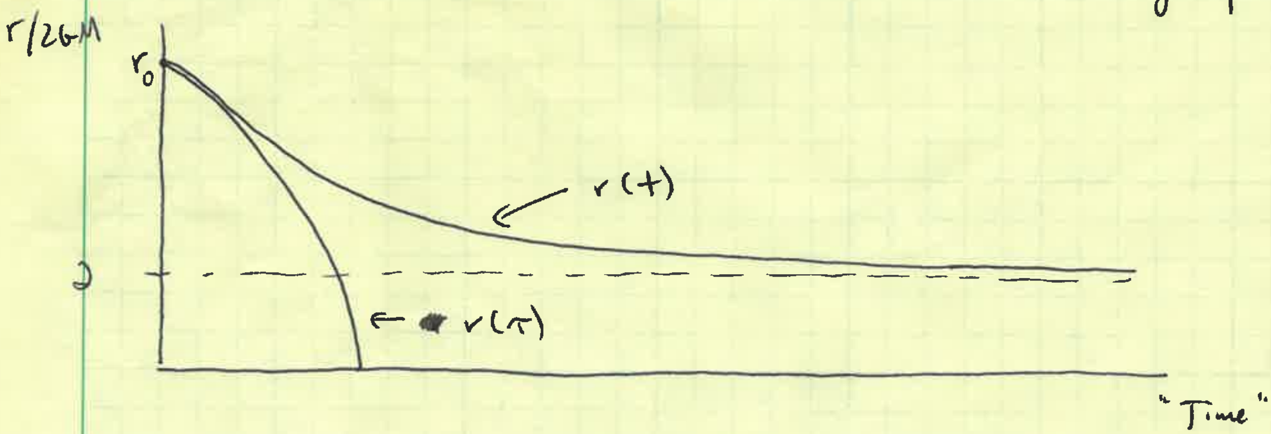
Result:

$$\frac{t}{2GM} = \ln\left[\frac{(r/2GM)^{1/2}+1}{(r/2GM)^{1/2}-1}\right] - 2\sqrt{\frac{r}{2GM}}\left(1 + \frac{r}{6GM}\right)$$

$$- \text{(same, replace } r \text{ with } r_0\text{)}$$

$$\frac{\tau}{2GM} = \frac{2}{3}\left[\left(\frac{r_0}{2GM}\right)^{3/2} - \left(\frac{r}{2GM}\right)^{3/2}\right]$$

Notice: $t \to \infty$ as $r \to 2GM$, but τ does nothing special.



Infalling body quickly plunges into $r = 0$ by its own clock ... but never crosses $r = 2GM$ according to distant observers.

Huh??