Kerr

$$ds^2 = -\left(1 - \frac{2GMr}{\rho^2}\right) dt^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2\, d\theta^2$$
$$+ \frac{\sin^2\theta}{\rho^2}\left[\left(r^2 + a^2\right)^2 - a^2 \Delta \sin^2\theta\right] d\phi^2$$
$$- \frac{4GMar\sin^2\theta}{\rho^2} dt\, d\phi$$

$$\rho^2 = r^2 + a^2 \cos^2\theta\ , \qquad \Delta = r^2 - 2GMr + a^2$$

Notice: independent of *t* and *φ*.

Constants of the motion

$$p_t = g_{tt}p^t + g_{t\phi}p^\phi \equiv -E \qquad p_\phi = g_{\phi t}p^t + g_{\phi\phi}p^\phi \equiv L_z$$

$$\longrightarrow E = -m\left(g_{tt}\frac{dt}{d\tau} + g_{t\phi}\frac{d\phi}{d\tau}\right) \quad \longrightarrow L_z = m\left(g_{\phi t}\frac{dt}{d\tau} + g_{\phi\phi}\frac{d\phi}{d\tau}\right)$$

Kerr geodesics also possess a “3rd constant” that was discovered by Brandon Carter using a relativistic Hamilton-Jacobi formalism; turns out to be related to a Killing tensor:

$$Q \equiv p_\theta^2 + \cos^2\theta\left[a^2(m^2 - E^2) + L_z^2/\sin^2\theta\right]$$
$$= m^2(d\theta/d\tau)^2 + \cos^2\theta\left[a^2(m^2 - E^2) + L_z^2/\sin^2\theta\right]$$

Conserved constants allows separation of equations of motion

$$\rho^4 \left(\frac{dr}{d\tau}\right)^2 = \left[E(r^2 + a^2) - aL_z\right]^2 - \Delta\left[r^2 + (L_z - aE)^2 + Q\right] \equiv R(r) \; ,$$

$$\rho^4 \left(\frac{d\theta}{d\tau}\right)^2 = Q - L_z^2 \cot^2\theta - a^2 \cos^2\theta[1 - E^2]$$

$$\rho^2 \left(\frac{d\phi}{d\tau}\right) = \csc^2\theta L_z + \frac{2MraE}{\Delta} - \frac{a^2 L_z}{\Delta}$$

$$\rho^2 \left(\frac{dt}{d\tau}\right) = E\left[\frac{(r^2 + a^2)^2}{\Delta} - a^2 \sin^2\theta\right] - \frac{2MraL_z}{\Delta}$$

Pick E, L_z, Q; determines motion.

Useful subset: $Q = 0$ means that $\theta = \pi/2$ for all time ("equatorial orbits")

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If we further have $R = 0$, $dR/dr = 0$ then we have defined *circular* equatorial orbits.

Characteristics of circular equatorial orbits

$$E = \frac{1 - 2v^2 \pm qv^3}{\sqrt{1 - 3v^2 \pm 2qv^3}} \qquad L_z = \pm rv \frac{1 \mp 2qv^3 + q^2v^4}{\sqrt{1 - 3v^2 \pm 2qv^3}}$$

$$v \equiv \sqrt{GM/r}\,, \qquad q \equiv a/M$$

$$\Omega = \frac{d\phi/d\tau}{dt/d\tau} = \pm \frac{\sqrt{GM}}{r^{3/2} \pm a\sqrt{GM}}$$

Top sign: Orbit is parallel to BH spin.

Bottom: Orbit is ***anti***parallel to spin.

Circular, equatorial orbit stability

Stable orbits have $d^2R/dr^2 < 0$ … metastable ones have $R = 0$, $dR/dr = 0$, $d^2R/dr^2 = 0$.

Apply this, find that it holds at radius

$$r_{\mathrm{ISCO}}/GM = 3 + Z_2 \mp \left[(3 - Z_1)(3 + Z_1 + 2Z_2)\right]^{1/2}$$

$$Z_1 = 1 + \left(1 - q^2\right)^{1/3}\left[(1+q)^{1/3} + (1-q)^{1/3}\right]$$

$$Z_2 = \left(3q^2 + Z_1^2\right)^{1/2}$$

Do exercise for zero mass particle:

$$r_{\mathrm{phot}}/GM = 2 + 2\cos\left[\frac{2}{3}\cos^{-1}(\mp q)\right]$$

(Reference: Bardeen, Press, Teukolsky, Astrophys. J. **178**, 347 (1972).

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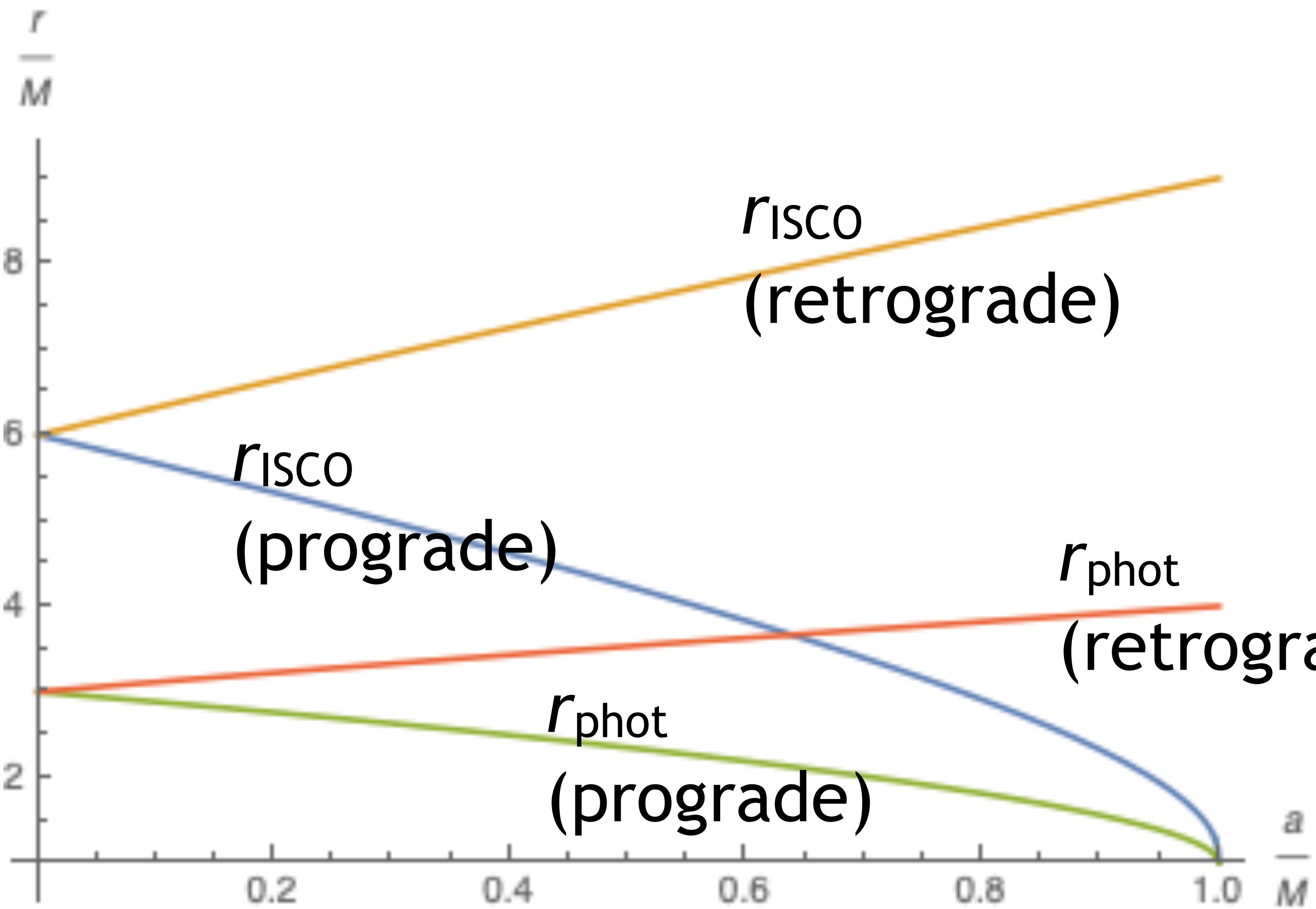

Frame dragging

Consider *non*-geodesic motion. Suppose I want to use a rocket or some other agent to hold myself fixed at a location, or to orbit with

$$\Omega_{\rm NG} = u^\phi / u^t \; ; \qquad u^r = 0 \quad u^\theta = 0$$

We must have $u_a u^a = -1$, which means

$$-1 = g_{tt}(u^t)^2 + 2 g_{t\phi} u^t u^\phi + g_{\phi\phi}(u^\phi)^2$$
$$= (u^t)^2 \left[g_{tt} + 2 g_{t\phi} \Omega_{\rm NG} + g_{\phi\phi} \Omega_{\rm NG}^2 \right]$$

Quantity in [] must be negative! Find $\Omega_{\rm NG}$ by solving quadratic.

Frame dragging

Solve quadratic:

$$\Omega_{\rm NG}^{\rm min/max} = \frac{-g_{t\phi} \pm \sqrt{g_{t\phi}^2 - g_{tt}g_{\phi\phi}}}{g_{\phi\phi}}$$

There is a radius at which g_{tt} goes to zero:

$$r_{\rm E}/GM = 1 + \sqrt{1 - q^2\cos^2\theta}$$

Coincides with event horizon at poles; is outside the horizon at all other θ.

No stationary observers exist for $r < r_{\rm E}$. They are dragged along by the spacetime, parallel to sense of the the hole's spin.

Frame dragging

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Region called the “ergosphere”: can use it to extract energy from BH spin (“Penrose process”).